Lesson 7: Trigonometry and Complex Numbers

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Problem of the Week

PotW

Consider sequence $\{a_i\}$ with first term $a_0 = \frac{13}{2}$ which satisfies $a_n = \lfloor a_{n-1} \rfloor + \frac{1}{a_{n-1}}$. Compute the numerator of $a_{2020} \pmod{1013}$.

- Note that $\lfloor a_n \rfloor = 6$ for all n.
- Rewrite the recursion as $a_n = 6 + \frac{1}{a_{n-1}} = \frac{6a_{n-1}+1}{a_{n-1}}$.
- Define b_n and c_n to be the numerator and denominator of a_n, respectively.

$$a_n = \frac{6a_{n-1} + 1}{a_{n-1}} = \frac{6b_{n-1} + c_{n-1}}{b_{n-1}}.$$

- Thus, $c_n = b_{n-1}$ and $b_n = 6b_{n-1} + c_{n-1} = 6b_{n-1} + b_{n-2}$. We have a recursion for the numerator!
- The characteristic polynomial of this recursion is $\lambda^2 - 6\lambda - 1$, which has roots $3 \pm \sqrt{10}$.

Problem of the Week

- We want to find $b_{2020} \pmod{1013}$, so we will work in mod 1013 for the rest of this solution.
- We claim there exists an a such that $a^2 \equiv 10 \pmod{1013}$ or that 10 is a quadratic residue. This is clear from quadratic reciprocity:

$$\left(\frac{10}{1013}\right) = \left(\frac{2}{1013}\right)\left(\frac{5}{1013}\right) = (-1)\left(\frac{1013}{5}\right) = (-1)\left(\frac{3}{5}\right) = 1$$

- This means we can write $3 + \sqrt{10} \equiv x \pmod{1013}$ and $3 - \sqrt{10} \equiv y \pmod{1013}$ for some integers x, y.
- The closed form of the recursion is $b_n \equiv c_1 x^n + c_2 y^n \pmod{1013}$ for integers c_1, c_2.
- By Fermat's little theorem,

$$b_{n+1012k} \equiv c_1 x^{n+1012k} + c_2 y^{n+1012k} = c_1 x^n + c_2 y^n \equiv b_n \pmod{1013}.$$

Problem of the Week

- This means that $b_{2020} = b_{-4}$, which we can easily calculate using the recursion $b_n = 6b_{n-1} + b_{n-2}$.
- We know $b_0 = 13$. The denominator of a_0 is b_{-1} from the work above, so $b_{-1} = 2$.
- We obtain $b_{-2} = 1$, $b_{-3} = -4$, and $b_{-4} = 25$, so $b_{2020} = 25 \pmod{1013}$.

Algebraic Trigonometry: Identity Review

- For algebraic trigonometry problems, it is usually necessary to know the basic trig identities.
- Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$.
- Addition and Subtraction Identities:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha.$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

Algebraic Trigonometry: More Identities

- Using the addition and subtraction formulas, we can also write double and half angle identities:

$$\sin 2x = 2\sin x\cos x,\ \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}.$$

$$\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1,\ \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}.$$

- Be careful with the plus or minus in the half angle identities.

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x},\ \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}.$$

2014 AMC 12B #25

Find the sum of all the positive solutions of

$$2\cos 2x\left(\cos 2x - \cos\left(\frac{2014\pi^2}{x}\right)\right) = \cos 4x - 1.$$

- We see $\cos 2x$ multiple times on the left side, so this motivates us to write the right side as a function of $\cos 2x$ with the double angle identity.
- $2\cos 2x\left(\cos 2x - \cos\left(\frac{2014\pi^2}{x}\right)\right) = \cos 4x - 1 = 2\cos^2 2x - 2.$
- Now, we can divide by 2 and expand the left side.

$$\cos^2 2x - \cos 2x \cos\left(\frac{2014\pi^2}{x}\right) = \cos^2 2x - 1.$$

$$\cos 2x \cos\left(\frac{2014\pi^2}{x}\right) = 1.$$

2014 AMC 12B #25

- From $\cos 2x \cos\left(\frac{2014\pi^2}{x}\right) = 1$, we must either have $\cos 2x = 1$ and $\cos\left(\frac{2014\pi^2}{x}\right) = 1$ or $\cos 2x = -1$ and $\cos\left(\frac{2014\pi^2}{x}\right) = -1$
- Suppose $\cos 2x = 1$ and $\cos\left(\frac{2014\pi^2}{x}\right) = 1$.
- This means that $2x = 2n\pi$ or $x = n\pi$ and $\frac{2014\pi^2}{x} = \frac{2014\pi}{n} = 2k\pi$ for integers n and k.
- This reduces to $nk = 1007$, so n can be any integer divisor of 1007, which would include 1, 19, 53, and 1007. The sum of the solutions is 1080π.
- Suppose $\cos 2x = -1$ and $\cos\left(\frac{2014\pi^2}{x}\right) = -1$.
- This means that $2x = (2n+1)\pi$, or $x = \left(n + \frac{1}{2}\right)\pi$.
- $\frac{2014\pi^2}{x} = \frac{2014\pi}{n+\frac{1}{2}} = \frac{4028\pi}{2n+1} = (2k+1)\pi$ for integers n and k. However, this implies $(2n+1)(2k+1) = 4028$, which cannot happen.
- So the sum of the solutions is 1080π.

Algebraic Trigonometry

AoPS

Find the value of the sum

$$\sum_{k=1}^{\infty} \arctan \frac{1}{2k^2}.$$

- Let's try to understand the addition of arctangents first.
- Start with the hypothetical question of adding $\arctan x + \arctan y$. Let $\arctan x + \arctan y = \theta$.
- Take a tangent of both sides:

$$\tan\theta = \tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \frac{x+y}{1-xy}$$

- Thus, $\arctan x + \arctan y = \theta = \arctan \frac{x+y}{1-xy}$

Infinite arctangent sum

- To understand our sum a little better, let's start computing partial sums using this identity for summing arctangents.
- $\sum_{k=1}^{1} \arctan \frac{1}{2k^2} = \arctan \frac{1}{2}$.
- $\sum_{k=1}^{2} \arctan \frac{1}{2k^2} = \arctan \frac{1}{2} + \arctan \frac{1}{8} = \arctan \frac{\frac{1}{2}+\frac{1}{8}}{1-\frac{1}{2}\cdot\frac{1}{8}} = \arctan \frac{2}{3}$.
- $\sum_{k=1}^{3} \arctan \frac{1}{2k^2} = \arctan \frac{2}{3} + \arctan \frac{1}{18} = \arctan \frac{\frac{2}{3}+\frac{1}{18}}{1-\frac{2}{3}\cdot\frac{1}{18}} = \arctan \frac{3}{4}$.
- It looks like we are starting to see a pattern, so we can form the following claim.

Claim

For all positive integers n,

$$\sum_{k=1}^{n} \arctan \frac{1}{2k^2} = \arctan \frac{n}{n+1}.$$

Infinite arctangent sum

- We can prove this claim with induction. The base case $n = 1$ has already been shown above. Now assume that $\sum_{k=1}^{m} \arctan \frac{1}{2k^2} = \arctan \frac{m}{m+1}$.
- We have

$$\begin{aligned}
\sum_{k=1}^{m+1} \arctan \frac{1}{2k^2} &= \sum_{k=1}^{m} \arctan \frac{1}{2k^2} + \arctan \frac{1}{2(m+1)^2} \\
&= \arctan \frac{m}{m+1} + \arctan \frac{1}{2(m+1)^2} \\
&= \arctan \frac{\frac{m}{m+1} + \frac{1}{2(m+1)^2}}{1 - \frac{m}{m+1}\frac{1}{2(m+1)^2}} = \arctan \frac{m+1}{m+2}
\end{aligned}$$

- This completes the induction. For the infinite sum, we know $\lim_{n\to\infty} \frac{n}{n+1} = 1$.
- Thus, $\sum_{k=1}^{\infty} \arctan \frac{1}{2k^2} = \arctan 1 = \frac{\pi}{4}$.

Sum-to-Product and Product-to-Sum

- Consider the sum $\sin(\alpha+\beta)+\sin(\alpha-\beta)$.
- By the addition and subtraction formulas, this is equal to $2\sin\alpha\cos\beta$.
- Thus, given the sum $\sin x+\sin y$, we can let $a=\frac{x+y}{2}$ and $b=\frac{x-y}{2}$ so that

$$\begin{aligned}\sin x+\sin y=\sin(a+b)+\sin(a-b)&=2\sin a\cos b\\&=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).\end{aligned}$$

- We can write the sum of two sines as a product of a sine and a cosine!

Sum-to-Product and Product-to-Sum Continued

- Here are all the **sum-to-product** and **product-to-sum** identities.

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right),$$

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right).$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)),$$

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)),$$

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta)).$$

Algebraic Trigonometry

COMC

Determine the sum of the angles A and B, where $0^\circ \le A, B \le 180^\circ$, and

$$\sin A + \sin B = \sqrt{\frac{3}{2}}, \qquad \cos A + \cos B = \sqrt{\frac{1}{2}}.$$

- We use sum-to-product, giving us $2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = \sqrt{\frac{3}{2}}$ and $2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = \sqrt{\frac{1}{2}}$.
- We note that we can cancel everything not involving $A+B$ by dividing the two equations, giving $\tan\left(\frac{A+B}{2}\right) = \sqrt{3}$
- Thus, $\frac{A+B}{2} = 60^\circ$, so $A+B = 120^\circ$.

AIME

Evaluate

$$\frac{\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \cdots + \cos 43^\circ + \cos 44^\circ}{\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots + \sin 43^\circ + \sin 44^\circ}.$$

- We note that the sums are difficult to deal, so we use sum-to-product.
- We pair up $\cos\theta + \cos(45^\circ - \theta) = 2\cos\left(\frac{45^\circ}{2}\right)\cos\left(\frac{45-2\theta}{2}\right)$ and $\sin\theta + \sin(45^\circ - \theta) = 2\sin\left(\frac{45^\circ}{2}\right)\cos\left(\frac{45-2\theta}{2}\right)$
- Now, note that repeatedly using this gives $\frac{2\cos\left(\frac{45^\circ}{2}\right)(\cos\left(\frac{1^\circ}{2}\right)+\cos\left(\frac{3^\circ}{2}\right)+\cdots+\cos\left(\frac{43^\circ}{2}\right))}{2\sin\left(\frac{45^\circ}{2}\right)(\cos\left(\frac{1^\circ}{2}\right)+\cos\left(\frac{3^\circ}{2}\right)+\cdots+\cos\left(\frac{43^\circ}{2}\right))}$
- Note that the second terms cancel, giving $\frac{\cos\left(\frac{45^\circ}{2}\right)}{\sin\left(\frac{45^\circ}{2}\right)}$
- $\frac{\cos\left(\frac{45^\circ}{2}\right)}{\sin\left(\frac{45^\circ}{2}\right)} = \frac{1}{\tan\left(\frac{45^\circ}{2}\right)} = \frac{1+\cos 45^\circ}{\sin 45^\circ} = \sqrt{2}+1$

Complex Numbers Review

- $z = a + bi$, a, b, real, $i^2 = -1$
- a is the *real part*, b is the *imaginary part*
- $\overline{z} = a - bi$ is the *conjugate* of z
- Basic arithmetic with complex numbers: adding and subtracting is direct; multiply using distributive property/FOIL, divide by multiplying numerator and denominator by the conjugate of the denominator:

$$\frac{3+4i}{1+2i} = \frac{(3+4i)(1-2i)}{(1+2i)(1-2i)} = \frac{11-2i}{5}$$

- Conjugate behaves nicely with respect to most operations:

$$\overline{a+b} = \overline{a} + \overline{b},\ \overline{a-b} = \overline{a} - \overline{b},\ \overline{ab} = \overline{a}\overline{b},\ \overline{\frac{a}{b}} = \frac{\overline{a}}{\overline{b}}$$

- If f is a polynomial with real coefficients then $\overline{f(z)} = f(\overline{z})$
- Most usefully if z is a root then $\overline{z}$ is a root

1995 AIME # 5

For certain real values of $a, b, c,$ and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$. Find b.

- Suppose $pq = 13 + i$; then p and q are not conjugates
- $\overline{p}$ and $\overline{q}$ must also be roots $\implies \overline{p} + \overline{q} = 3 + 4i$
- Now $\overline{pq} = 13 - i$, $p + q = 3 - 4i$ by properties of the conjugate
- By Vieta $b = pq + p\overline{p} + q\overline{q} + p\overline{q} + q\overline{p} + \overline{pq}$
- Factor $pq + \overline{pq} + (p + q)(\overline{p} + \overline{q})$
- This is $(13 + i) + (13 - i) + (3 - 4i)(3 + 4i) = \boxed{51}$

Polar Form of Complex Numbers

- Complex numbers can also be described in terms of their *magnitude* and *argument*
- $|z| = |a + bi| = \sqrt{a^2 + b^2}$ is the distance from the origin
- $\arg z$ is angle between the line connecting the origin and z in the complex plane and the positive x-axis

$$\arg(a + bi) = \begin{cases} \arctan\left(\frac{b}{a}\right) & a + bi \text{ in quadrants I or IV} \\ \arctan\left(\frac{b}{a}\right) + 180^\circ & a + bi \text{ in quadrants II or III} \end{cases}$$

- Extremely useful properties of magnitude and argument:

$$|ab| = |a||b|, \ \arg(ab) = \arg(a) + \arg(b)$$

- De Moivre's Theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
- The n solutions to $x^n = 1$ are $\cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$ where $0 \le k \le n - 1$ and are called the *nth roots of unity*

2005 AIME II # 9

For how many positive integers n less than or equal to 1000 is

$$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$

true for all real t?

- LHS is $(\cos(90 - t) + i\sin(90 - t))^n$
- By De Moivre this is $\cos(90 - t)n + i\sin(90 - t)n$
- Now need $\cos(90 - t)n = \sin nt,\ \sin(90 - t)n = \cos nt$
- This is $\cos(90n - nt) = \sin nt,\ \sin(90n - nt) = \cos nt$
- First equation is $\cos 90n\cos nt + \sin 90n\sin nt = \sin nt$
- Need $\cos 90n = 0, \sin 90n = 1$
- Second equation gives the same thing
- So $n \equiv 1 \pmod 4 \implies \boxed{250}$

Polar Form of Complex Numbers

2000 AIME II # 9

Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3^\circ$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

- Solve for z: $z + \frac{1}{z} = 2\cos 3^\circ \implies z^2 - 2\cos 3^\circ z + 1 = 0$
- $z = \frac{2\cos 3^\circ \pm \sqrt{(2\cos 3^\circ)^2 - 4}}{2} = \cos 3^\circ \pm i \sin 3^\circ = \cos 3^\circ + i \sin \pm 3^\circ$
- Now we can use De Moivre! $z^{2000} = \cos 6000^\circ + i \sin \pm 6000^\circ$
- This is $z^{2000} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$
- So $z^{2000} + \frac{1}{z^{2000}} = \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right) + \left(-\frac{1}{2} \mp \frac{\sqrt{3}}{2} i\right) = -1$
- $\boxed{0}$

Polar Form of Complex Numbers

2008 AIME I # 8

Find the positive integer n such that

$$\arctan\frac{1}{3}+\arctan\frac{1}{4}+\arctan\frac{1}{5}+\arctan\frac{1}{n}=\frac{\pi}{4}.$$

- Why are complex numbers useful?
- Easy to deal with tangents: $\arg(3+i)=\arctan\frac{1}{3}$
- Multiply numbers to add arguments:
- $\arg((3+i)(4+i))=\arctan\frac{1}{3}+\arctan\frac{1}{4}$
- Now $\arg((3+i)(4+i)(5+i)(n+i))=\frac{\pi}{4}$
- $(3+i)(4+i)(5+i)(n+i)=(48n-46)+(46n+48)i$
- The argument of this is $\frac{\pi}{4}\implies 48n-46=46n+48$
- $n=\boxed{47}$