Lesson 5: Summations B

Adithya B., Brian L., William W., Daniel X.

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POTW

YCMA POTW

Consider the sum $S = \sum_{k=0}^{2020} \frac{4^k}{\binom{2k}{k}}$. If it can be written in simplest form as $\frac{a}{b}$, compute the last 3 digits of $3a - b$.

- These terms seem like they are related to the ones around them, so what if we tried to take advantage of that.
- $\frac{4^{k+1}}{\binom{2k+2}{k+1}} = \frac{4(k+1)^2}{(2k+1)(2k+2)} \frac{4^k}{\binom{2k}{k}} = \frac{2k+2}{2k+1} \frac{4^k}{\binom{2k}{k}}$
- If we could get this to telescope, that'd be great. Suppose that $\frac{4^k}{\binom{2k}{k}} = f(k+1) - f(k)$ for some k.
- $f(k)$ probably shares a lot of factors with $\frac{4^k}{\binom{2k}{k}}$, so suppose that $f(k) = \frac{4^k}{\binom{2k}{k}} g(k)$

POTW

- Then, after simplification, we find that we need $1 = \frac{2k+2}{2k+1} g(k+1) - g(k)$.
- Seeing that we probably want $g(k+1) = a(2k+1)$, or $g(k) = a(2k-1)$ for some a, we plug this in to get $1 = a(2k+2) - a(2k-1)$, so $a = \frac{1}{3}$.
- Thus, we find that $g(k) = \frac{1}{3}(2k-1)$ works, so $f(k) = \frac{1}{3}(2k-1)\frac{4^k}{\binom{2k}{k}}$ works.
- Now, we just wish to calculate $f(2021) - f(0)$
- $f(2021) - f(0) = \frac{4041}{3}\frac{4^{2021}}{\binom{4042}{2021}} + \frac{1}{3}$. Recall that we want $3a - b$, which happens to be the numerator of $\frac{3a-b}{3b} = \frac{a}{b} - \frac{1}{3} = 1347\frac{4^{2021}}{\binom{4042}{2021}}$.

Combinatorial Arguments

Vandermonde's Identity

For nonnegative integers m, n, r,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

- LHS is easy to interpret; what does the RHS mean?
- $\binom{m}{k}$: number of ways to choose k stones from m stones
- $\binom{n}{r-k}$: number of ways to choose $r-k$ stones from n stones.
- $\binom{m}{k}\binom{n}{r-k}$: number of ways to choose k red stones and $r-k$ blue stones from m red stones and n blue stones
- If we sum over all possible k, we get the number of ways to choose r stones from m red stones and n blue stones
- This is the LHS!

More Vandermonde

2017 AIME I # 7

For nonnegative integers a and b with $a+b \leq 6$, let $T(a,b) = \binom{6}{a}\binom{6}{b}\binom{6}{a+b}$. Let S denote the sum of all $T(a,b)$, where a and b are nonnegative integers with $a+b \leq 6$. Find the remainder when S is divided by 1000.

- $T(a,b)$ looks similar to the product from Vandermonde
- Choose a red stones from 6 red stones, b blue stones from 6 blue stones, $a+b$ green stones from 6 green stones
- Or choose a red stones from 6 red stones, b blue stones from 6 blue stones, $6-a-b$ green stones from 6 green stones
- Sum over all a, b: number of ways to choose 6 stones of *any* color from 6 red, 6 blue, 6 green stones
- $\binom{18}{6}$, answer 564

Combinatorial Arguments

China TST 2014/2/1

Prove that for any positive integers k and N,

$$\left(\frac{1}{N}\sum_{n=1}^{N}(\omega(n))^k\right)^{\frac{1}{k}} \leq k + \sum_{q\leq N}\frac{1}{q},$$

where $\sum_{q\leq N}\frac{1}{q}$ is the summation over all prime powers $q \leq N$ (including $q = 1$).

Note: For an integer $n > 1$, $\omega(n)$ denotes number of distinct prime factors of n, and $\omega(1) = 0$.

China TST 2014/2/1

$$\left(\frac{1}{N}\sum_{n=1}^{N}(\omega(n))^k\right)^{\frac{1}{k}} \leq k + \sum_{q \leq N}\frac{1}{q},$$

- Rewrite as $\frac{1}{N}\sum_{n=1}^{N}(\omega(n))^k \leq \left(k + \sum_{q \leq N}\frac{1}{q}\right)^k$
- $(\omega(n))^k$ is the number of k-tuples of primes dividing n, want to find the number of k-tuples for each $1 \leq n \leq N$
- Sum swap: we want to find the number of n for each k-tuple
- Whether a particular n is valid for a given k-tuple depends only on the set of distinct primes in the k-tuple
- If our k tuple has distinct primes $\{q_1, \ldots, q_i\}$, the number of possible n is $\lfloor \frac{N}{q_1 \cdots q_i} \rfloor$

China TST 2014/2/1

- The number of possible k-tuples for the given $\{q_1, \ldots, q_i\}$ is hard to count (i.e. number of tuples $(x_1, \ldots, x_k)$ which have exactly this set of primes); let's upper bound it as follows
- There are $\binom{k-1}{i-1}$ ways to choose how many times each q_i appears by stars and bars
- Given how many times each q_i appears, we have at most $k!$ ways to permute them
- So there are at most $k!\binom{k-1}{i-1}$ k-tuples
- The sum on the left is at most

$$\frac{1}{N}\sum_{\{q_1,\ldots,q_i\}}\left\lfloor\frac{N}{q_1\cdots q_i}\right\rfloor k!\binom{k-1}{i-1} \leq \sum_{\{q_1,\ldots,q_i\},q_j\leq N}\frac{1}{q_1\cdots q_i}k!\binom{k-1}{i-1}$$

- Upon expansion, the RHS also has a term of the form $\frac{1}{q_1\cdots q_i}$. What is its coefficient? Remember, the RHS is $\left(k+\sum_{q\leq N}\frac{1}{q}\right)^k$

China TST 2014/2/1

- The RHS contains the term $\frac{k^{k-i}}{q_1 \dots q_i} \cdot \binom{k}{i} i! = \frac{k^{k-i}}{q_1 \dots q_i} \cdot \frac{k!}{(k-i)!}$
- We would be done if the coefficient of $\frac{1}{q_1 \cdots q_i}$ on the RHS was greater than that of the LHS. Let's try to verify this
- This is

$$\frac{1}{q_1 \cdots q_i} k! \binom{k-1}{i-1} \leq \frac{k^{k-i}}{q_1 \cdots q_i} \cdot \frac{k!}{(k-i)!}$$

- Divide by $\frac{k!}{q_1 \cdots q_i}$ and we need

$$\binom{k-1}{i-1} \leq \frac{k^{k-i}}{(k-i)!}$$

- Which is

$$\frac{(k-1)!}{(i-1)!(k-i)!} \leq \frac{k^{k-i}}{(k-i)!} \iff \frac{(k-1)!}{(i-1)!} \leq k^{k-i}$$

which is true as this is $(k-1)(k-2) \cdots i \leq k^{k-i}$

Generating Functions

- Given a sequence $a_0, a_1, a_2, \ldots$, the **generating function** of the sequence is

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i.$$

- We can use a generating function to find the term a_i in the sequence by determining the coefficient of x^i.
- A simple generating function is

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

- Another common generating function is

$$\frac{1}{(1-x)^k} = \binom{k-1}{k-1} + \binom{k}{k-1}x + \binom{k+1}{k-1}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{k-1+i}{k-1} x^i.$$

- Try to prove this with induction on k!

2007 HMMT C9

Let $S = \{(i, j, k) \mid i + j + k = 17, i, j, k \geq 1\}$. Find

$$\sum_{(i,j,k) \in S} ijk.$$

- Note that we are are given i, j, and k have a fixed sum of 17.
- Therefore, this motivates us to look at the coefficient of x^{17} in the generating function for the sequence with terms $a_n = \sum_{i+j+k=n} ijk$.
- Note that $\frac{1}{(1-x)^2} = \sum_{i \geq 0} (i+1)x^i$, and therefore $\frac{x}{(1-x)^2} = \sum_{i \geq 1} ix^i$.
- If we multiply the following generating functions, the coefficient of x^{17} is precisely the sum we want:

$$\left(\sum_{i \geq 0} ix^i\right)\left(\sum_{j \geq 0} jx^j\right)\left(\sum_{k \geq 0} kx^k\right) = \sum_{n \geq 0}\left(\sum_{i+j+k=n} ijk\right)x^n$$

2007 HMMT C9

- Therefore, the generating function we want is $\left(\frac{x}{(1-x)^2}\right)^3 = \frac{x^3}{(1-x)^6}$.
- We can expand this:

$$\frac{x^3}{(1-x)^6} = x^3\left(\binom{5}{5} + \binom{6}{5}x + \binom{7}{5}x^2 + \cdots\right)$$

- We want the coefficient of x^{17}, which is $\binom{19}{5}$.

Generating Functions

Classical

The Catalan Numbers are defined by $C_0 = 1$ and

$$C_n = C_{n-1}C_0 + C_{n-2}C_1 + \ldots + C_0C_{n-1}$$

Find x such that $C_0 + C_1x + C_2x^2 + \ldots = \frac{7}{4}$.

- Denote the generating function for the Catalan numbers by $C(x)$ and note that $C_0 + C_1x + C_2x^2 + \cdots$ is precisely the generating function for the Catalan numbers.
- The recursion for C_n contains all the terms C_iC_j where $i + j = n - 1$.
- If we multiply $C(x)$ by itself, then we can group all of these terms in the recursion together.
- Note that

$$(C_0+C_1x+C_2x^2+\cdots)^2 = C_0^2+(C_0C_1+C_1C_0)x+(C_0C_2+C_1^2+C_2C_0)x^2+\cdots$$

Catalan Generating Function (Clasical)

- Now, note that $C_0^2 = C_1$, $(C_0C_1 + C_1C_0) = C_2$, etc. Therefore,

$$\begin{aligned} C(x)^2 &= C_1 + C_2x + C_3x^2 + \cdots \\ &= \frac{(C_0 + C_1x + C_2x^2 + \cdots) - C_0}{x} \\ &= \frac{C(x) - 1}{x} \end{aligned}$$

- Nice!, we have an equation with only $C(x)$ and x. We can rewrite this as $xC(x)^2 - C(x) + 1 = 0$.
- For this question, we are given that $C(x) = \frac{7}{4}$, so we can plug this in to get $\frac{49}{16}x - \frac{7}{4} + 1 = 0 \implies x = \frac{12}{49}$.
- Now, for completion, we can compute an expression for the Catalan generating function with the quadratic formula:

$$C(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

- We take the negative root because as $x \to 0$, $C(x) \to 1$.

Snake Oil

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1.
2.
3.
4.
5.

Snake Oil

Well Known

For $n \geq 0$, compute
$$\sum_{k\geq 0} \binom{2k}{k}\binom{n}{k}\left(-\frac{1}{4}\right)^k.$$

- We notice that n is our free variable here, so we can form a generating function for this sum that has the values for this sum at each value of n.

$$\sum_{n\geq 0}\left(\sum_{k\geq 0} \binom{2k}{k}\binom{n}{k}\left(-\frac{1}{4}\right)^k\right)x^n = \sum_{n\geq 0}\sum_{k\geq 0} \binom{2k}{k}\binom{n}{k}\left(-\frac{1}{4}\right)^k x^n$$

$$= \sum_{k\geq 0}\sum_{n\geq 0} \binom{2k}{k}\binom{n}{k}\left(-\frac{1}{4}\right)^k x^n = \sum_{k\geq 0} \binom{2k}{k}\left(-\frac{1}{4}\right)^k \sum_{n\geq 0}\binom{n}{k} x^n$$

Snake Oil

- Note that $\sum_{n\geq 0}\binom{n}{k}x^n = \frac{x^k}{(1-x)^{k+1}}$.

$$\sum_{k\geq 0}\binom{2k}{k}\left(-\frac{1}{4}\right)^k\sum_{n\geq 0}\binom{n}{k}x^n = \sum_{k\geq 0}\binom{2k}{k}\left(-\frac{1}{4}\right)^k\frac{x^k}{(1-x)^{k+1}}$$
$$= \frac{1}{1-x}\sum_{k\geq 0}\binom{2k}{k}\left(\frac{-\frac{1}{4}x}{1-x}\right)^k$$

Lemma

The generating function $\sum_{k\geq 0}\binom{2k}{k}y^k = \frac{1}{\sqrt{1-4y}}$.

Proof.

Multiply the Catalan generating function by x and take a derivative. Details are in the handout. □

Snake Oil

- Use the lemma!

$$\frac{1}{1-x}\sum_{k\geq 0}\binom{2k}{k}\left(\frac{-\frac{1}{4}x}{1-x}\right)^k = \frac{1}{1-x}\frac{1}{\sqrt{1-4\cdot\left(\frac{-\frac{1}{4}x}{1-x}\right)}}$$

$$= \frac{1}{1-x}\frac{1}{\sqrt{1+\frac{x}{1-x}}} = \frac{1}{1-x}\frac{1}{\sqrt{\frac{1}{1-x}}}$$

$$= \frac{1}{1-x}\sqrt{1-x} = \frac{1}{\sqrt{1-x}}$$

- But from the lemma where $y = \frac{x}{4}$, we have $\frac{1}{\sqrt{1-x}} = \sum_{n\geq 0}\binom{2n}{n}\left(\frac{x}{4}\right)^n = \sum_{n\geq 0}\frac{\binom{2n}{n}}{4^n}x^n$. So the sum is $\frac{\binom{2n}{n}}{4^n}$.

Number Theoretic Summations

Some sums you encounter may involve the use of number theoretic functions, so it is always helpful to be familiar with the more common ones. We review a few here:

- $\lfloor \cdot \rfloor$: Floor function is the largest integer $\leq n$. In number theoretic sums, they typically appear as $\left\lfloor \frac{n}{k} \right\rfloor$. In this context, they count the number of multiples of k less than or equal to n.
- $\varphi(n)$: Known as Euler's Totient Function, this counts the number of positive integers $\leq n$ which are relatively prime to it. For $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, we have that $\varphi(n) = \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right) \cdot n$
- $\tau(n)$ or $d(n)$: The number of divisors of n. For $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, we have $\tau(n) = (1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_k)$ This is because we have $1 + \alpha_i$ possible choices for the exponent of p_i for each i
- $\sigma(n)$: The sum of the divisors of n. For $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, we have $\sigma(n) = (1 + p_1 + \dots p_1^{\alpha_1}) \cdots (1 + p_k + \dots + p_k^{\alpha_k}) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \dots \cdot \frac{p_k^{\alpha_k+1}-1}{p_k-1}$

Number Theoretic Summations

2019 CMIMC Algebra #7

For all positive integers n, let

$$f(n) = \sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor^2 .$$

Compute $f(2019) - f(2018)$. Here $\varphi(n)$ denotes the number of positive integers less than or equal to n which are relatively prime to n.

- We wish to find the difference between 2 consecutive values of $f(n)$.
- Consider the terms that change between $f(n)$ and $f(n-1)$
- Notice that only the terms that have k as a divisor of n change, since otherwise, $\left\lfloor \frac{n}{k} \right\rfloor$ is the same for n and $n-1$
- Now, we note that
  $f(n) - f(n-1) = \sum_{k|n} \varphi(k)(\frac{n}{k})^2 - (\frac{n}{k} - 1)^2 = \sum_{k|n} \varphi(k)(2\frac{n}{k} - 1)$

Number Theoretic Summations

- Now, we split this into 2 sums, $2n\sum_{k|n}\varphi(k)\frac{1}{k}$ and $-\sum_{k|n}\varphi(k)$.
- Now, we factor this by prime divisor, giving $2n\prod_{p|n}(1+(1-\frac{1}{p})e)$ and $\prod_{p|n}(1+\sum_{i=1}^{e}(p-1)p^{i-1})$.
- Calculating this for $n = 2019$ gives $13450 - 2019 = 11431$

Dirichlet Convolution

Dirichlet Convolution is an extremely powerful technique which can help trivialize many otherwise difficult NT summations.

Multiplicativity

For a function $f : \mathbb{N} \to \mathbb{R}$, we call it multiplicative if $f(x)f(y) = f(xy)$ for all pairs x, y with $\gcd(x, y) = 1$

Convolution

For two functions, $f, g : \mathbb{N} \to \mathbb{R}$, we define their dirichlet convolution to be

$$f * g(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right)$$

Ex: $f * g(4) = f(1)g(4) + f(2)g(2) + f(4)g(1)$

Dirichlet Convolution

Theorem

Given multiplicative functions f, g, we have that $f * g$ is also multiplicative.

- Denote $c = f * g$. We want to show $c(xy) = c(x)c(y)$ for $\gcd(x, y) = 1$
- $c(x)c(y) = \left(\sum_{d|x} f(d)g\left(\frac{x}{d}\right)\right)\left(\sum_{d'|y} f(d')g\left(\frac{y}{d'}\right)\right)$
- Try expanding. Each term is of the form

$$f(d)f(d')g\left(\frac{x}{d}\right)g\left(\frac{y}{d'}\right) = f(dd')g\left(\frac{xy}{dd'}\right)$$

- We can combine terms this way because $\gcd(x, y) = 1$, so the gcd of a divisor of x and a divisor of y is also 1
- Now, note that every $k|xy$ can be expressed uniquely as $k = dd'$ where $d|x$, $d'|y$
- Thus, $c(x)c(y) = c(xy)$ as desired

Dirichlet Convolution

2015 PUMaC NT #6

For a positive integer n, let $d(n)$ be the number of positive divisors of n. What is the smallest positive integer n such that

$$\sum_{t|n} d(t)^3$$

is divisible by 35?

- Is $d(t)$ multiplicative? How about $d(t)^3$?
- This is a convolution of $d(t)^3$ and 1. So, if we let $f(n) = \sum_{t|n} d(t)^3$, we have that f is multiplicative
- Why is multiplicativity important? This means that if the prime factorization of n is $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then

$$f(n) = f(p_1^{\alpha_1}) \cdots f(p_k^{\alpha_k})$$

2015 PUMaC NT #6

- So, we only have to consider f when n is a prime power. What happens when $n = p^k$?
- $f(p^k) = d(1)^3 + d(p)^3 + \ldots + d(p^k)^3 = 1^3 + 2^3 + \ldots + (k+1)^3$, which is just $\left(\frac{(k+1)(k+2)}{2}\right)^2$
- So, $f(n) = \left(\frac{(\alpha_1+1)(\alpha_1+2)}{2}\right)^2 \cdots \left(\frac{(\alpha_k+1)(\alpha_k+2)}{2}\right)^2$. We want one factor to be divisible by 5 and one to be divisible by 7 (not 35 because otherwise n gets too big)
- So, we have $\alpha_1 = 3$, $\alpha_2 = 5$
- Minimum n is $2^5 \cdot 3^3 = 864$

Dirichlet Convolution

Bulgaria 1989

Let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity. Evaluate

$$\sum_{n=1}^{1989} (-1)^{\Omega(n)} \left\lfloor \frac{1989}{n} \right\rfloor$$

- Verify that $(-1)^{\Omega(n)}$ is multiplicative
- Unfortunately, this isn't in convolution form. How do we make it in convolution form?
- Recall that $\left\lfloor \frac{1989}{n} \right\rfloor$ is the number of multiples of n less than or equal to 1989, so we can write

$$\left\lfloor \frac{1989}{n} \right\rfloor = \sum_{n|k, k \leq 1989} 1$$

Bulgaria 1989

- Now, we have a sum inside a sum, what do we do?
- Sum swap! We get

$$\sum_{n=1}^{1989}(-1)^{\Omega(n)} \sum_{n|k,k\leq 1989} 1 = \sum_{k=1}^{1989}\sum_{n|k}(-1)^{\Omega(n)}$$

- Aha! The inner term is a convolution! If we define $f(n) = \sum_{n|k}(-1)^{\Omega(n)}$, what is $f(p^k)$?
- $f(p^k) = 1 - 1 + 1 - \ldots + (-1)^k$, so it is 1 if $2|k$ and 0 otherwise
- This means that, $f(n) = 1$ if and only if all the exponents of the primes dividing it are even, and it is 0 otherwise. What does this tell you about n?
- This is exactly the definition of a square! The inside is 1 if n is a square, and 0 otherwise!
- So, our answer is the number of squares under 1989, or $\boxed{44}$