Lesson 4: Summations A

Adithya B., Brian L., William W., Daniel X.

May 2020

Sample POTW

YCMA Sample POTW

Let x,y,z be real numbers such that

$$x^2+y^2+z^2-\frac{1000}{9}=xy+xz+yz \text{ and } x^2+y=y^2+z=z^2+x$$

If the maximum possible value of $x+y$ can be written as $\frac{a+\sqrt{b}}{c}$, for naturals a,b,c where c is minimal, compute $a+b+c$.

- Note that we have the factorization $x^2+y^2+z^2-xy-xz-yz=\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right)$.
- The first equation can be rearranged to $(x-y)^2+(y-z)^2+(z-x)^2=\frac{2000}{9}$.
- If we rearrange $x^2+y=y^2+z$, then we get $x^2-y^2=z-y$ or $(x-y)(x+y)=-(y-z)$.
- Seeing the expressions $x-y$ and $y-z$ again, we are motivated to substitute $a=x-y$, $b=y-z$, and $c=z-x$.

YCMA Sample POTW

- Note that $a+b+c=0$ and $a^2+b^2+c^2=\frac{2000}{9}$. From this, we can obtain $ab+bc+ac=-\frac{1000}{9}$.
- The equation $(x-y)(x+y)=-(y-z)$ can be rewritten as $x+y=-\frac{b}{a}$.
- Similarly, we can use $y^2+z=z^2+x$ to obtain $y+z=-\frac{c}{b}$.
- Subtract these two equations: $(x+y)-(y+z)=\frac{c}{b}-\frac{b}{a}=-c$.
- Multiply this by ab to get $ac-b^2=-abc$.
- Now, we can use the other equations (or use symmetry) to argue that $ab-c^2=-abc$ and $bc-a^2=-abc$.
- Adding these three equations, $-3abc=ab+bc+ac-a^2-b^2-c^2=3(ab+bc+ac)-(a+b+c)^2=-\frac{3000}{9}$. Therefore, $abc=\frac{1000}{9}$.
- Now, from Vieta's a,b,c are roots of the equation $t^3-\frac{1000}{9}t-\frac{1000}{9}$. By factoring/solving this cubic, the roots are $-10, 5\pm\frac{5\sqrt{13}}{3}$.
- Now, remember that $x+y=-\frac{b}{a}$, so $x+y=-\frac{5+\frac{5\sqrt{13}}{3}}{5-\frac{5\sqrt{13}}{3}}=\frac{11+\sqrt{117}}{2}$.

Basic Sums

- Notation: $\sum_{i=m}^{n} f(i)$ means $f(m) + f(m+1) + \ldots + f(n)$
- Basic formulae:
  - Sum of first n terms of arithmetic sequence $a, a+d, \ldots = \frac{n(2a+(n-1)d)}{2}$
  - Sum of first n terms of geometric sequence $a, ar, \ldots = a\frac{r^n-1}{r-1}$
  - Sum of infinite geometric sequence $a, ar, \ldots$ with $|r| < 1 = \frac{a}{1-r}$

Basic Sums

2010 MPFG #16

Let P be the quadratic function such that $P(0) = 7$, $P(1) = 10$, and $P(2) = 25$. If a, b, and c are integers such that every positive number x less than 1 satisfies

$$\sum_{n=0}^{\infty} P(n)x^n = \frac{ax^2 + bx + c}{(1-x)^3},$$

compute the ordered triple (a, b, c).

- First find the polynomial: $P(x) = 6x^2 - 3x + 7$
- Our sum is

$$\sum_{n=0}^{\infty} P(n)x^n = 6\sum_{n=0}^{\infty} n^2x^n - 3\sum_{n=0}^{\infty} nx^n + 7\sum_{n=0}^{\infty} x^n$$

2010 MPFG #16

- The last sum is normal infinite geometric: $\frac{1}{1-x}$
- If we let $S = \sum_{n=0}^{\infty} nx^n$, compute xS
- $xS = \sum_{n=1}^{\infty} (n-1)x^n$, so

$$S(1-x) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$$

- $S = \frac{x}{(1-x)^2}$
- Now, let $T = \sum_{n=0}^{\infty} n^2 x^n$. Let's try the same thing by computing xT
- $xT = \sum_{n=1}^{\infty} (n-1)^2 x^n$, so

$$(1-x)T = \sum_{n=1}^{\infty} (2n-1)x^n$$

2010 MPFG #16

- We know the right hand side!

$$\sum_{n=1}^{\infty}(2n-1)x^n = 2\sum_{n=1}^{\infty} nx^n - \sum_{n=1}^{\infty} x^n = 2S - \frac{x}{1-x}$$

- So,

$$T = \frac{1}{1-x}\left(\frac{2x}{(1-x)^2} - \frac{x}{1-x}\right) = \frac{x^2+x}{(1-x)^3}$$

- Putting everything together, the sum is $\frac{16x^2-11x+7}{(1-x)^3}$

Telescoping

- For a sum $\sum_{i=1}^{n} f(i)$, $f(i)$ can sometimes be written as $f(i) = g(i+1) - g(i)$ for some function g
- This allows us to cancel a lot of terms, as $g(i), -g(i)$ appear once each for $2 \leq i \leq n$, meaning $\sum_{i=1}^{n} f(i) = g(n+1) - g(1)$
- The sum collapses on itself like a telescope, hence its name
- To spot telescoping series, be on the lookout for a neat way to write $f(i)$ as a difference
    - Partial fractions
    - Trying to relate consecutive terms
    - Sum to product

Telescoping

2002 AIME I #4

Consider the sequence defined by $a_k = \frac{1}{k^2+k}$ for $k \geq 1$. Given that $a_m + a_{m+1} + \cdots + a_{n-1} = 1/29$, for positive integers m and n with $m < n$, find $m + n$.

- Split a_k with partial fractions: $a_k = \frac{1}{k} - \frac{1}{k+1}$
- The sum telescopes as $\frac{1}{m} - \frac{1}{n}$
- We want $\frac{1}{m} - \frac{1}{n} = \frac{1}{29} \implies 29(n-m) = mn$
- This factors as $(29-m)(29+n) = 841$
- $n = 812, m = 28$, so $m + n = 840$

Telescoping

MOP

Compute the sum
$$\sum_{k=0}^{n} \frac{(4k+1)k!}{(2k+1)!}$$

- Try to relate parts of the consecutive terms. Factorials are generally easy to connect, so look at $\frac{k!}{(2k+1)!}$ and $\frac{(k+1)!}{(2k+3)!}$
- We have that $\frac{k!}{(2k+1)!} = (4k+6) \cdot \frac{(k+1)!}{(2k+3)!}$
- Find a way to substitute this back into the initial summand
- Note that $\frac{(4k+1)k!}{(2k+1)!} = \frac{(4k+2)k!}{(2k+1)!} - \frac{k!}{(2k+1)!} = \frac{(4k+2)k!}{(2k+1)!} - \frac{(4k+6)(k+1)!}{(2k+3)!}$
- This telescopes!
- If we plug in $k+1$ for k into $\frac{(4k+2)k!}{(2k+1)!}$ we will get $\frac{(4k+6)(k+1)!}{(2k+3)!}$

MOP

- So, if we let $f(k) = \frac{(4k+2)k!}{(2k+1)!} = \frac{(k-1)!}{(2k-1)!}$, our final answer is $f(0) - f(n+1)$
- $\sum_{k=0}^{n} \frac{(4k+1)k!}{(2k+1)!} = 2 - \frac{k!}{(2k+1)!}$

Double Sums

- Sometimes we are asked to compute a sum whose individual terms are also sums
- e.g. $\sum_{a=0}^{\infty}\left(\sum_{b=0}^{\infty}\frac{1}{2^{a+b}}\right)$; how do we evaluate this?
- The inner sum is $\frac{1}{2^a}+\frac{1}{2^{a+1}}+\frac{1}{2^{a+2}}+\cdots=\frac{1}{2^{a-1}}$.
- Thus the outer sum is

$$\frac{1}{2^{-1}}+\frac{1}{2^0}+\frac{1}{2^1}+\cdots=4.$$

- A good way to think of double sums is as the sum of all of the numbers in a square table:

$$\begin{pmatrix} 2^0 & 2^{-1} & 2^{-2} & \cdots & \\ 2^{-1} & 2^{-2} & 2^{-3} & \cdots & \\ 2^{-2} & 2^{-3} & 2^{-4} & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Double Sums

- There are a couple of strategies that can help evaluate double sums:
- Most importantly, *being flexible with the order of summation.* For example, we can write the sum we had earlier as

  $$(2^0)+(2^{-1}+2^{-1})+(2^{-2}+2^{-2}+2^{-2})+(2^{-3}+2^{-3}+2^{-3}+2^{-3})+\cdots$$

  by summing along diagonals.
- We can also sometimes *factor* double (or multiple) sums into the product of multiple single sums. An example is the sum of divisors formula. If we wanted to find the sum of the divisors of $1000000 = 2^6 5^6$ we could write

  $$\sum_{i=0}^{6}\sum_{j=0}^{6} 2^i 5^j = \left(\sum_{i=0}^{6} 2^i\right)\left(\sum_{j=0}^{6} 5^j\right)$$

Double Sums

2017 HMMT Algebra and Number Theory #5

Kelvin the Frog was bored in math class one day, so he wrote all ordered triples (a, b, c) of positive integers such that $abc = 2310$ on a sheet of paper. Find the sum of all integers he wrote down. In other words, compute

$$\sum_{\substack{abc=2310 \\ a,b,c\in\mathbb{N}}} (a+b+c),$$

where $\mathbb{N}$ denotes the positive integers.

- Want the sum of all numbers that Kelvin writes down
- How many times does a given number n appear? (Clearly n has to be a divisor of 2310).
- 3 times the number of times Kelvin writes down n as the first number in the triple

Double Sums

- If $a = n$ then $bc = \frac{2310}{n}$
- $\tau\left(\frac{2310}{n}\right)$ ways to choose (b, c)
- Thus Kelvin writes down n as the first number $\tau\left(\frac{2310}{n}\right)$ times
- So Kelvin writes down n a total of $3\tau\left(\frac{2310}{n}\right)$ times
- The answer is thus

$$3 \sum_{n|2310} n \cdot \tau\left(\frac{2310}{n}\right)$$

- $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
- Sum can be written as

$$32 + 16(2 + 3 + 5 + 7 + 11) + 8(2 \cdot 3 + \cdots + 7 \cdot 11) + \cdots + 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$$

- This equals $(2+2)(3+2)(5+2)(7+2)(11+2) = 16380$
- Final answer $3 \cdot 16380 = 49140$

Double Sums

2017 PUMaC Algebra A #7

Compute

$$\sum_{k=0}^{\infty} \frac{2^k}{5^{2^k}+1}.$$

- At first glance, this seems like a nasty single sum that doesn't simplify. How can we handle the terms?
- All denominators are of the form $1+$ something; what can we do with this?
- Infinite geometric series expansion:

$$\frac{1}{1+5^{2^k}} = \frac{1}{5^{2^k}} - \frac{1}{5^{2\cdot 2^k}} + \frac{1}{5^{3\cdot 2^k}} - \frac{1}{5^{4\cdot 2^k}} + \frac{1}{5^{5\cdot 2^k}} - \cdots$$

2017 PUMaC Algebra A #7

- Do this to every term: we get

$$\begin{aligned}
\frac{1}{1+5^1} &= \frac{1}{5^1} - \frac{1}{5^2} + \frac{1}{5^3} - \frac{1}{5^4} + \frac{1}{5^5} - \frac{1}{5^6} + \frac{1}{5^7} - \frac{1}{5^8} + \cdots \\
\frac{2}{1+5^2} &= \frac{2}{5^2} - \frac{2}{5^4} + \frac{2}{5^6} - \frac{2}{5^8} + \cdots \\
\frac{4}{1+5^4} &= \frac{4}{5^4} - \frac{4}{5^8} + \cdots \\
\frac{8}{1+5^8} &= \frac{8}{5^8} - \cdots
\end{aligned}$$

This is $\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots$!

If 2^k is the largest power of 2 dividing n then the coefficient of 5^{-n} is

$$-1 - 2 - 4 - \cdots - 2^{k-1} + 2^k = 1$$

Final sum $\frac{1}{4}$

Double Sums

2017 HMMT Algebra and Number Theory #2

Find the value of

$$\sum_{1 \le a < b < c} \frac{1}{2^a 3^b 5^c}$$

(i.e. the sum of $\frac{1}{2^a 3^b 5^c}$ over all triples of positive integers (a, b, c) satisfying $a < b < c$).

- We notice that we can write all triples (a, b, c) with $1 \le a < b < c$ as triples $(a, a + x, a + x + y)$ with $a, x, y \ge 1$
- Thus, we write the sum as $\sum_{a=1}^{\infty} \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \frac{1}{2^a 3^{a+x} 5^{a+x+y}}$
- We simplify this into $\sum_{a=1}^{\infty} \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \frac{1}{30^a 15^x 5^y}$.

Double Sums

- We now note that this can be factored into
$$\left(\sum_{a=1}^{\infty} \frac{1}{30^a}\right)\left(\sum_{x=1}^{\infty} \frac{1}{15^x}\right)\left(\sum_{y=1}^{\infty} \frac{1}{5^y}\right) = \frac{\frac{1}{30}}{1-\frac{1}{30}} \frac{\frac{1}{15}}{1-\frac{1}{15}} \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{1}{1624}$$

Double Sums

2013 HMMT Algebra #7

Compute
$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1+a_2+\cdots+a_7}{3^{a_1+a_2+\cdots+a_7}}.$$

- Split into seven terms, each of the form $\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1}{3^{a_1+a_2+\cdots+a_7}}$.
- Factor into $\left(\sum_{a_1=0}^{\infty}\frac{a_1}{3^{a_1}}\right)\left(\sum_{a_2=0}^{\infty}\frac{1}{3^{a_2}}\right)\left(\sum_{a_3=0}^{\infty}\frac{1}{3^{a_3}}\right)\cdots\left(\sum_{a_7=0}^{\infty}\frac{1}{3^{a_7}}\right)$.
- Each term after the first is equal to $\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$.
- The first one is equal to $\frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^2}=\frac{3}{4}$.
- Thus, we find that our answer is $7\left(\frac{3}{4}\right)\left(\frac{3}{2}\right)^6=\frac{15309}{256}$

Ad hoc

APMO 2000/1

Compute the sum: $\sum_{i=0}^{101} \frac{x_i^3}{1-3x_i+3x_i^2}$ for $x_i = \frac{i}{101}$.

- Notice that the denominator looks sort of like $(1-x_i)^3$.
- The denominator is $(1-x_i)^3 + x_i^3 = x_{101-i}^3 + x_i^3$. Therefore, the sum is $\sum_{i=0}^{101} \frac{x_i^3}{x_{101-i}^3+x_i^3}$.
- But now notice that $\frac{x_i^3}{x_{101-i}^3+x_i^3} + \frac{x_{101-i}^3}{x_{101-i}^3+x_i^3} = 1$. Therefore, we can pair up the terms so that they add to 1.
- The sum becomes

$$\sum_{i=0}^{101} \frac{x_i^3}{x_{101-i}^3 + x_i^3} = \sum_{i=0}^{50} \left(\frac{x_i^3}{x_{101-i}^3 + x_i^3} + \frac{x_{101-i}^3}{x_{101-i}^3 + x_i^3} \right) = 51.$$

Ad hoc

2007 AIME I # 11

For each positive integer p, let $b(p)$ denote the unique positive integer k such that $|k - \sqrt{p}| < \frac{1}{2}$. For example, $b(6) = 2$ and $b(23) = 5$. If $S = \sum_{p=1}^{2007} b(p)$, find the remainder when S is divided by 1000.

- The given inequality is $k - \frac{1}{2} < \sqrt{p} < k + \frac{1}{2}$.
- We can square this relation to get rid of the square root: $k^2 - k + \frac{1}{4} < p < k^2 + k + \frac{1}{4}$.
- Since k is a positive integer, we have that $b(p) = k$ for $p = k^2 - k + 1, k^2 - k + 2, \ldots, k^2 + k$. This is a total of $2k$ integers.
- Note that the largest possible integer with $b(p) = 44$ is $44^2 + 44 = 1980$. Therefore, from $p = 1981$ to $p = 2007$, $b(p) = 45$. There are $2007 - 1981 + 1 = 27$ numbers such that $b(p) = 45$.
- Therefore, $\sum_{p=1}^{2007} b(p) = \sum_{k=1}^{44} k \cdot 2k + 27 \cdot 45$.
- Using $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, the sum is $2 \cdot \frac{44 \cdot 45 \cdot 89}{6} + 27 \cdot 45 = 59955$.