Lesson 6: Sequences

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Problem of the Week

PotW

Define

$$f(n) = \sum_{i=0}^{\infty} \frac{\gcd(n, i) \cdot i}{x^i}$$

If $f(n) = \frac{P_n(x)}{Q_n(x)}$ for integer polynomials P_n, Q_n which don't share any common factors other than ± 1, find the minimal positive integer n such that $2019 | P_n(1)$

- This sum at first glance looks pretty intractable. In particular, we want to get rid of $\gcd(n, i)$
- We can try splitting the sum into sums based on gcd. For example, we want to write it as

$$f(n) = \sum_{d|n} \sum_{i=0}^{\infty} \frac{d \cdot (di)}{x^{di}}$$

Problem of the Week

- This is incorrect! The coefficient of $\frac{(i)}{x^i}$ is now $\sum_{d|\gcd(n,i)} d$ instead of $\gcd(n,i)$.
- However, recall that $\sum_{d|\gcd(n,i)} \phi(d) = \gcd(n,i)$, so if we write $f(n) = \sum_{i=0}^{\infty} \frac{i}{x^i} \sum_{d|\gcd(n,i)} \phi(d)$ we can now sumswap as

$$\sum_{d|n} \phi(d) \sum_{i=0}^{\infty} \frac{(di)}{x^{di}}$$

- Recall $\sum_{i=0}^{\infty} \frac{i}{x^i} = \frac{x}{(x-1)^2}$, so this inside is $\frac{dx^d}{(x^d-1)^2}$
- So, $f(n) = \sum_{d|n} \frac{d\phi(d)x^d}{(x^d-1)^2}$. What should $Q_n(x)$ be?

Problem of the Week

- All denominators go into $(x^n - 1)^2$, so we should have $Q_n(x) = (x^n - 1)^2$
- This means that

$$P_n(x) = \sum_{d|n} \frac{d\phi(d)x^d(x^n - 1)^2}{(x^d - 1)^2} = \sum_{d|n} d\phi(d)x^d(1 + x^d + \ldots + x^{n-d})^2$$

- When we plug in 1, we get

$$P_n(x) = \sum_{d|n} d\phi(d)\left(\frac{n}{d}\right)^2 = n\sum_{d|n} \phi(d)\frac{n}{d}$$

- The sum is a convolution! So, we only need to consider prime powers

Problem of the Week

- $\sum_{d|p^k} \phi(d)\frac{n}{d} = p^k + p^{k-1}(p-1) + p^{k-2} \cdot p(p-1) + \ldots$, which is $p^{k-1}(p + k(p-1))$
- So, if $q|P_n(1)$, we either have $q|n$ or $q|(p+k(p-1))$ for some $p|n$.
- Now, we are ready to finish. If $2019|P_n(1)$, we need $3, 673|P_n(1)$. How can we get 3?
- We either choose $3|n$ or $3|(p+k(p-1))$. $(p,k) = (2,1)$ works and $2^1 < 3$
- For 673, if $673|(p+k(p-1))$, then we should have $(k+1)(p-1) > 673$. This basically shows that if $k > 1$, then p^k will be at least $\left(\frac{673}{k+1}\right)^k$ which is much too large. On the other hand, $k = 1$ admits $(p,k) = (337,1)$ which is small enough
- Our answer is $n = 2 \cdot 337 = \boxed{674}$

Basic Sequences

- An *arithmetic* sequence is a sequence in which the difference between any two consecutive terms is a constant
- $a, a+d, a+2d, a+3d, \ldots$
- a is the first term, d is the *common difference*
- Each term is the average of the two adjacent terms
- A *geometric* sequence is a sequence in which the ratio between any consecutive two terms is a (nonzero) constant
- $a, ar, ar^2, ar^3, \ldots$
- a is the first term, r is the *common ratio*
- Each term is the geometric mean of the two adjacent terms
- It is very useful to write arithmetic/geometric sequences in terms of two parameters: one term and the common difference/ratio

2003 AIME I #8

In an increasing sequence of four positive integers, the first three terms form an arithmetic progression, the last three terms form a geometric progression, and the first and fourth terms differ by 30. Find the sum of the four terms.

- Let's write the first three terms as $a-d, a, a+d$ instead of $a, a+d, a+2d$
- This makes the fourth term $\frac{(a+d)^2}{a}$:

$$a-d, a, a+d, \frac{(a+d)^2}{a}$$

- Now we can write $\frac{(a+d)^2}{a} - (a-d) = 30$

2003 AIME I #8

- We can simplify: $(a+d)^2 - a(a-d) = 30a$
- $d^2 + 3ad = 30a$
- Let's solve for a in terms of d because this is a linear equation:
- $a = \frac{d^2}{30-3d}$
- We know a, d must be positive integers, so $1 \leq d \leq 9$
- We can narrow down d more: since $30 - 3d$ is divisible by 3 we know that d^2 is divisible by 3
- So $d \in \{3, 6, 9\}$
- $d = 3 \implies a = \frac{3}{7}$
- $d = 6 \implies a = 3$; but $a - d < 0$ so this doesn't work
- $d = 9 \implies a = 27$; gives

$$18, 27, 36, 48$$

- So answer is $18 + 27 + 36 + 48 = 129$

Basic Sequences

2004 AIME II #9

A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $n \geq 1$, the terms a_{2n-1}, a_{2n}, a_{2n+1} are in geometric progression, and the terms a_{2n}, a_{2n+1}, and a_{2n+2} are in arithmetic progression. Let a_n be the greatest term in this sequence that is less than 1000. Find $n + a_n$.

- Not completely arithmetic or geometric, but we can write $a_1 = 1, a_2 = r, a_3 = r^2$
- a_2, a_3, a_4 in arithmetic progression so $a_4 = 2a_3 - a_2 = 2r^2 - r$
- a_3, a_4, a_5 in geometric progression so $a_5 = \frac{a_4^2}{a_3} = \frac{(2r^2-r)^2}{r^2} = (2r-1)^2$
- a_4, a_5, a_6 in arithmetic progression so $a_6 = 2a_5 - a_4 = 2(2r-1)^2 - r(2r-1) = (3r-2)(2r-1)$

2004 AIME II #9

- a_5, a_6, a_7 in geometric progression so $a_7 = \frac{a_6^2}{a_5} = \frac{(3r-2)^2(2r-1)^2}{(2r-1)^2} = (3r-2)^2$
- Can continue:

$$a_8 = (4r-3)(3r-2)$$
$$a_9 = (4r-3)^2$$
$$a_{10} = (5r-4)(4r-3)$$

- We know $a_9 + a_{10} = 646$ so $(4r-3)^2 + (5r-4)(4r-3) = 646$
- Solve this to get the positive solution $r = 5$
- Now we can calculate all terms of the sequence: $1, 5, 25, 45, 81, \ldots$
- Can get that $a_{16} = 957$ is largest term less than 1000 so answer $16 + 957 = 973$
- Recommend trying the calculations on your own; good exercise in keeping computations neat and expressions factored

Recursions

- Broadly speaking, recursions are sequences where the nth term is defined by previous ones, such as $x_n = x_{n-1} + \ldots + x_0$ or $x_n = \frac{1}{x_{n-1}} + 1$
- As they are a very broad class of sequences, there is no particular global method. However, keep an eye out for
  - Periodicity: Check if a recursive equation might actually be hiding periodicity, such as $x_n = \frac{1}{x_{n-1}}$ for an easy example
  - Large scale behavior: See how the sequence behaves as a whole
  - Generating functions: Many sequences are susceptible to be solved using generating functions, in a method similar to Snake Oil

2012 AIME I #11

Let $f_1(x) = \frac{2}{3} - \frac{3}{3x+1}$, and for $n \geq 2$, define $f_n(x) = f_1(f_{n-1}(x))$. The value of x that satisfies $f_{1001}(x) = x - 3$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

- To get a better feel for the sequence, let's write out a few of the terms and see if we notice anything special.
- Note that

$$f_1(x) = \frac{2}{3} - \frac{3}{3x+1},$$
$$f_2(x) = \frac{2}{3} - \frac{3}{3(\frac{2}{3} - \frac{3}{3x+1}) + 1} = \frac{2}{3} - \frac{3x+1}{3x-2},$$
$$f_3(x) = \frac{2}{3} - \frac{3}{3(\frac{2}{3} - \frac{3x+1}{3x-2}) + 1} = x.$$

- The sequence is periodic!

2012 AIME I #11

- Then, $f_4(x) = f_1(x)$, so we have the period is 3.
- $f_{1001}(x) = f_2(x) = \frac{2}{3} - \frac{3x+1}{3x-2} = x - 3$.
- Multiply both sides by $3(3x-2)$:

$$2(3x-2) - 3(3x+1) = 3(3x-2)(x-3)$$

$$-3x - 7 = 9x^2 - 33x + 18 \implies 9x^2 - 30x + 25 = 0$$

- This factors as $(3x-5)^2$, so $x = \frac{5}{3}$. The answer is 008.

Recursions

2018 PUMaC Algebra #7

Let the sequence $\{a_n\}_{n=-2}^{\infty}$ satisfy $a_{-1} = a_{-2} = 0$, $a_0 = 1$, and for all non-negative integers n,

$$n^2 = \sum_{k=0}^{n} a_{n-k} a_{k-1} + \sum_{k=0}^{n} a_{n-k} a_{k-2}.$$

Given a_{2018} is rational, find the maximum integer m such that 2^m divide the denominator of the reduced form of a_{2018}.

- The sums look sort of similar to the Catalan number recursion: $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$.
- Let's try generating functions. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$.
- The sum in the question can be written as $n^2 = \sum_{k=0}^{n} a_{n-k}(a_{k-1} + a_{k-2})$.
- Let's define $b_k = a_{k-1} + a_{k-2}$ so that $n^2 = \sum_{k=0}^{n} a_{n-k} b_k$.

- Define the generating function for b_i to be $B(x)$. Let's find $A(x)B(x)$.

$$(a_0+a_1x^2+a_2x^2+\cdots)(b_0+b_1x+b_2x^2+\cdots)$$
$$=a_0b_0+(a_0b_1+a_1b_0)x+(a_0b_2+a_1b_1+a_2b_0)x^2+\cdots$$
$$=0^2+1^2\cdot x+2^2\cdot x^2+\cdots=\sum_{n=0}^{\infty}n^2x^n$$

- Remember the following:

$$\frac{2}{(1-x)^3}=\sum_{n=0}^{\infty}2\binom{n+2}{2}x^n=\sum_{n=0}^{\infty}(n^2+3n+2)x^n.$$
$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}\binom{n+1}{1}x^n=\sum_{n=0}^{\infty}(n+1)x^n$$
$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

- Therefore,

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x} = \frac{x+x^2}{(1-x)^3} = A(x)B(x)$$

- Now, our goal is to find $A(x)$, so we want to find a way to relate $A(x)$ and $B(x)$.

$$\begin{aligned} B(x) = \sum_{n=0}^{\infty}(a_{n-1}+a_{n-2})x^n &= x\sum_{n=0}^{\infty} a_{n-1}x^{n-1} + x^2\sum_{n=0}^{\infty} a_{n-2}x^{n-2} \\ &= (x+x^2)A(x) \end{aligned}$$

- Therefore, $A(x)B(x) = (x+x^2)A(x)^2 = \frac{x+x^2}{(1-x)^3}$, so

$$A(x) = \frac{1}{(1-x)^{\frac{3}{2}}}.$$

2018 PUMaC Algebra #7

Lemma

$$\frac{1}{(1-x)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} \frac{2n+1}{4^n} \binom{2n}{n} x^n.$$

Proof.

Try to expand with the binomial theorem! □

- Note that $a_{2018} = \frac{4037}{4^{2018}} \binom{4036}{2018}$.
- Note that

$$\nu_2\left(\binom{4036}{2018}\right) = \nu_2(4036!) - 2\nu_2(2018!) = 4029 - 2 \cdot 2011 = 7.$$

- There are 4036 powers of 2 in the denominator and 7 in the numerator, so the largest power in the denominator after simplifying is 2^{4029}.

Linear Recursions

- A linear recurrence is one of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

- Linear recurrences appear a lot, and one of the most important steps towards understanding them is to find a closed form.
- Let's use ideas from last week on generating functions to try to find a closed form!

Linear Recursions

- Consider $S = \sum_{i=0}^{\infty} a_i x^i$. What is $c_1 x S + c_2 x^2 S + \ldots + c_k x^k S$?
- The coefficient of x^n for $n \geq k$ in this expression is $c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} = a_n$. So, it is around S. However, we aren't guaranteed that x^n for $n < k$ has the right coefficient, so we actually have

$$c_1 x S + \ldots + c_k x^k S = S - P(x)$$

  for $P(x)$ with degree $< k$
- So, solving for S,

$$S = \frac{P(x)}{1 - c_1 x - c_2 x^2 - \ldots - c_k x^k}$$

- Suppose that the polynomial in the denominator has roots $r_1, \ldots, r_k$

Linear Recursions

- The denominator is $-c_k(x-r_1)(x-r_2)\dots(x-r_k)$. So, we can use partial fraction decomposition to get

$$S = \frac{m_1 r_1}{r_1 - x} + \frac{m_2 r_2}{r_2 - x} + \dots + \frac{m_k r_k}{r_k - x}$$

(for now we assume that all r_i are distinct.)

- Now, if we write each term as its own generating function, remember $\frac{r_1}{r_1 - x} = \sum_{i=0}^{\infty} \left(\frac{x}{r_1}\right)^i$, so we will get

$$S = \sum_{n=0}^{\infty} x^n \sum_{i=1}^{k} \left(\frac{1}{r_i}\right)^n m_i$$

- Equating coefficients, $a_n = \alpha_1^n m_1 + \alpha_2^n m_2 + \dots + \alpha_k^n m_k$ where $\alpha_i = \frac{1}{r_i}$

Linear Recursions

- $\alpha_1, \ldots, \alpha_k$ are the roots of $C(x) = x^n - c_1 x^{n-1} - \ldots - c_k$. We call this the characteristic polynomial
- The constants $m_1, m_2, \ldots, m_k$ can be solved for given the initial conditions
- Now, we will address root multiplicity. If a root has multiplicity 2, for example, it will appear in the partial fraction decomposition as $\frac{m+bx}{(r-x)^2}$ instead of $\frac{m}{r-x}$. In the generating function, this will manifest as $(p+qn)\alpha^n$ for some p, q. This idea generalizes, so if a root has multiplicity d, then the closed form has $P(n)\alpha^n$ where $P(n)$ is a polynomial in n of degree $d-1$.
- Ex: If $C(x)$ has roots $2, 2, 3$, the general form will be $(a+bn)2^n + c3^n$

Brilliant

A sequence x_n is defined by $x_0 = -1$, $x_1 = 0$, $x_2 = 1$ and the recurrence relation

$$x_n = 6x_{n-1} - 12x_{n-2} + 8x_{n-3}.$$

Find the closed form of x_n.

- This is a linear recurrence! Using our new terminology, what is its characteristic polynomial?
- Its characteristic polynomial is $x^3 - 6x^2 + 12x - 8$
- Notice that this can be factored into $(x-2)^3$
- Thus, we see that we can write x_n as $P(n)2^n$ for some degree 2 polynomial P
- Notice that we need $P(0) = -1$, $P(1) = 0$, $P(2) = \frac{1}{4}$.
- We write $P(n) = an^2 + bn + c$ and solve the resulting system.
- We get $a = -\frac{3}{8}$, $b = \frac{11}{8}$, $c = -1$, so the closed form is $x_n = (-\frac{3}{8}n^2 + \frac{11}{8}n - 1)2^n$

Linear Recursions

1990 AIME #15

Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

- We've already done this question before. How can we apply our knowledge of linear recurrence to it?
- Notice that these terms look like consecutive terms of a linear recurrence with characteristic polynomial with roots at x, y, so $\lambda^2 - (x + y)\lambda + xy$. Call this sequence $t_n = ax^n + by^n$
- Suppose we let $u = x + y, v = -xy$. Then, we have $t_n = ut_{n-1} + vt_{n-2}$

1990 AIME #15

- Now, we have $7u + 3v = 16$, $16u + 7v = 42$.
- Solving the system gives $u = -14, v = 38$.
- Thus, we find that the next term in the sequence, $t_5 = ax^5 + by^5 = -14 \cdot 42 + 38 \cdot 16 = 20$.

Weird Sequences

- Unfortunately, most sequences elude easy characterization
- The majority of sequences are "weird" and require ad hoc methods in order to solve
- The ideas to solve such questions are similar to those for general recursions:
  - Try to compute small terms if possible. Guess a pattern through engineers induction and try to prove it
  - Look at how the sequence behaves as a whole, and if there are any overarching global patterns
  - Be on the look out for manipulations, such as factorizations and substitutions which will simplify how the sequence looks
  - If initial conditions are given, see if they are special by trying the question with your own conditions. This will tell you if they are significant or if you can replace them with variables
  - Try to prove subresults to get a better intuition with the sequence. Jot down ideas you may have or qualities of the sequence you see, even if they have little to do with what we are actually trying to prove

Weird Sequences

2017 CMIMC Algebra & Number Theory #9

Define a sequence $\{a_n\}_{n=1}^{\infty}$ via $a_1 = 1$ and $a_{n+1} = a_n + \lfloor \sqrt{a_n} \rfloor$ for all $n \geq 1$. What is the smallest N such that $a_N > 2017$?

- Let's first try to get a feel for the sequence. What happens when we start at $a_k = n^2$?
- After 2 moves, we are at $a_{k+2} = n^2 + 2n = (n+1)^2 - 1$, and after a 3rd, we get to $a_{k+3} = (n+1)^2 + (n-1)$
- How about if we start at $a_k = n^2 + i$ with $0 < i < n$?
- After two turns we get to $a_{k+2} = (n+1)^2 + (i-1)$
- After a square n^2, it takes three turns to increase $\lfloor \sqrt{a_k} \rfloor$ by 1, and we overshoot the next square by $n-1$. After a nonsquare, we increase $\lfloor \sqrt{a_k} \rfloor$ after 2 turns, and our "overshoot" decreases by 1

2017 CMIMC Algebra & Number Theory #9

- So, if we start at a square n^2, it will take 3 moves to get to $(n+1)^2 + (n-1)$, and $2 * (n-1)$ more to get to the next square at $(2n)^2$.
- This tells us that the only squares in the sequence are $1, 4, 16, 64, \ldots$, and 4^{n+1} occurs $2(2^n - 1) + 3 = 2^{n+1} + 1$ terms after 4^n
- As $a_1 = 4^0$, 4^n occurs at position $1 + (2+1) + (4+1) + \ldots + (2^n + 1) = 2^{n+1} + (n-1)$
- So, $a_{2^6+4} = a_{68} = 1024$
- By previous logic, $a_{71} = (32+1)^2 + 31$, and after each pair of terms, the argument of the square increases by 1 while the outside "overshoot" decreases by 1
- We want to get close to 45^2, which is 12 away from 33. So, add 24 more terms
- $a_{95} = 45^2 + (31 - 12) = 2025 + 19$

2017 CMIMC Algebra & Number Theory #9

- $a_{94} = 2025 + 19 - 44 < 2017$, so our answer is $\boxed{95}$

Weird Sequences

Romania TST 2003/1

Let $(a_n)_{n\geq 1}$ be a sequence for real numbers given by $a_1 = 1/2$ and for each positive integer n

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}.$$

Prove that for every positive integer n we have $a_1 + a_2 + \cdots + a_n < 1$.

- The recurrence doesn't look very nice. How can we simplify it?
- Substituting $b_n = \frac{1}{a_n}$, we get $b_{n+1} = b_n^2 - b_n + 1$
- Now, it's easy to compute terms of $\{b_i\}$. We have $b_1 = 2$, $b_2 = 3$, $b_3 = 7$, $b_4 = 43$, $b_5 = 1807$. Any patterns?
- We see $b_{n+1} = b_1 b_2 \cdots b_n + 1$

Romania TST 2003/1

- To prove this, rearrange our equality as $\frac{b_{n+1}-1}{b_n-1} = b_n$. Now, if we multiply this telescopes
- $b_1 b_2 \cdots b_n = \frac{b_{n+1}-1}{b_1-1} = b_{n+1} - 1$
- Now, we can substitute a_{n+1} back in. We get that

$$a_{n+1} = \frac{1}{1 + b_1 b_2 \cdots b_n} = \frac{a_1 a_2 \cdots a_n}{1 + a_1 a_2 \cdots a_n}$$

- We now get

$$\frac{a_1 \cdots a_n}{1 + a_1 \cdots a_n} = (a_1 \cdots a_n) - \frac{(a_1 \cdots a_n)^2}{1 + a_1 \cdots a_n} = (a_1 \cdots a_n) - (a_1 \cdots a_n a_{n+1})$$

- $a_1 + a_2 + \ldots + a_n$ telescopes! It becomes $1 - a_1 a_2 \cdots a_n$, so we are done.