Lesson 1: Polynomials

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Definitions

- $P(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$
- z is a root of $P(x)$ iff $P(z) = 0$
- In the above P, we denote n as the degree of the polynomial, or $\deg P$
- Fundamental Theorem of Algebra: Counted with multiplicity, a polynomial $P(x)$ has exactly $\deg P$ roots, all of which are complex
- If P has roots $z_1, \ldots, z_n$, then it can always be factored as

$$P(x) = c_n(x - z_1)(x - z_2)\ldots(x - z_n)$$

Vieta's Formulas

- $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = a_n(x - r_1)(x - r_2) \cdots (x - r_n)$
- Suppose we expand the right hand side and compare the coefficients of x^k for each $0 \leq k \leq n - 1$
- The equations we get are called *Vieta's Formulas*:

Results

- $r_1 + r_2 + \ldots + r_n = -\frac{a_{n-1}}{a_n}$
- $r_1 r_2 + r_1 r_3 + \ldots + r_{n-1} r_n = \frac{a_{n-2}}{a_n}$.
- $\ldots$
- $r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}$
- The sum of all possible products of k roots is equal to $(-1)^k \frac{a_{n-k}}{a_n}$

Vieta's Example

Ex: $P(x) = (x - r_1)(x - r_2)(x - r_3)$
$P(x) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - r_1 r_2 r_3$

Vieta Problems

2008 AIME I # 7

Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

- Expand $(r+s)^3 + (s+t)^3 + (t+r)^3$
- $r+s+t=0$, $(r+s)^3 = -t^3$
- $-(r^3+s^3+t^3)$. No need to expand.
- $8r^3 + 1001r + 2008 = 0 \implies -r^3 = \frac{1001r+2008}{8}$
- $-(r^3+s^3+t^3) = \frac{1001(r+s+t)+2008*3}{8} = \frac{2008*3}{8} = 753$
- $r^3+s^3+t^3 = (r+s+t)(r^2+s^2+t^2-rs-st-rt)+3rst$

Vieta Problems

2013 HMMT Algebra #5

Let a and b be real numbers, and let r,s, and t be the roots of $f(x) = x^3 + ax^2 + bx - 1$. Also, $g(x) = x^3 + mx^2 + nx + p$ has roots r^2,s^2, and t^2. if $g(-1) = -5$, find the maximum possible value of b.

- Express m, n, p in terms a, b
- $p = -r^2s^2t^2$. But, $rst = 1$, so $p = -1$
- $m = -r^2 - s^2 - t^2$.
  $r^2 + s^2 + t^2 = (r + s + t)^2 - 2(rs + st + rt) = a^2 - 2b$
- $b^2 = r^2s^2 + r^2t^2 + s^2t^2 + 2rst(r + s + t) \implies n = b^2 + 2a$
- $g = -1 + m - n - 1 = -2 + 2b - a^2 - b^2 - 2a = -5 \implies$
  $a^2 + 2a + (b^2 - 2b - 3) = 0$
- $(a + 1)^2 + b^2 - 2b - 4 = 0 \implies b^2 - 2b - 4 \leq 0 \implies b \leq 1 + \sqrt{5}$

Factorization Problems

2016 AIME I #11

Let $P(x)$ be a nonzero polynomial such that $(x-1)P(x+1) = (x+2)P(x)$ for every real x, and $(P(2))^2 = P(3)$. Then $P(\frac{7}{2}) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

- Plug in $x = -2, 1$. Plugging in -2 gives $P(-1) = 0$ and 1 gives $P(1) = 0$, so $-1, 1$ are roots of $P(x)$.
- Now, try $x = -1$. We get $RHS = P(-1) = 0$, so 0 is also a root.
- So, we can factor $P(x)$ as $P(x) = (x-1)x(x+1)Q(x)$
- Now, plug this form back into the original equation. We get $Q(x+1) = Q(x)$, after canceling out $(x-1)x(x+1)(x+2)$
- Now, what can Q be? $Q(x) - Q(0)$ has roots $0, 1, 2, 3, \ldots$, so it must be the 0 polynomial, and $Q(x) = c$ for some c

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- $P(2) = 6c$, $P(3) = 24c$, so $c = 2/3$
- Now, we have $P(x) = \frac{2}{3}x(x-1)(x+1)$. Plugging in $7/2$, our final answer is $105/4$

Factorization

2014 USAMO/1

Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

- Vieta's formulas will work, but they get ugly fast
- If we plug in z into $P(x)$, we get $(z - x_1)(z - x_2)(z - x_3)(z - x_4)$. How can we get the squares?
- $x^2 + 1$ is actually a difference of squares! $x^2 + 1 = (x - i)(x + i)$
- Now, the product becomes $(x_1 + i) \cdots (x_4 + i) \cdot (x_1 - i) \cdots (x_4 - i) = P(i)P(-i)$
- Now, plug back into the original polynomial. We get $P(i)P(-i) = (1 + ai - b - ci + d)(1 - ai - b + ci + d)$, which is $(1 - b + d + (a - c)i)(1 - b + d - (a - c)i) = (1 - b + d)^2 + (a - c)^2$

Factorization

2014 USAMO/1

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- Now, plug back into the original polynomial. We get $P(i)P(-i) = (1 + ai - b - ci + d)(1 - ai - b + ci + d)$, which is $(1 - b + d + (a - c)i)(1 - b + d - (a - c)i) = (1 - b + d)^2 + (a - c)^2$
- Now, $b - d \geq 5$, so we get $(5 - 1)^2 = 16$.

Remainder Theorem

- Remainder Theorem: If P has real coefficients, the remainder when $P(x)$ is divided by $x - c$ is $P(c)$.

Proof.

- $P(x)$ can be written as $(x - c)Q(x) + R(x)$, where $Q(x)$ is the quotient and $R(x)$ Is the remainder.
- $\deg R < \deg(x - c) = 1$.
- Therefore, $R(x)$ is a constant function.
- Plugging in $x = c$ yields $P(c) = R(c)$, so the remainder is $P(c)$.

□

Remainder Theorem Problems

AoPS Intermediate Algebra

Find the remainder when $x^{100} - 4x^{98} + 5x + 6$ is divided by $x^3 - 2x^2 - x + 2$.

- The cubic is factorable as $(x-2)(x-1)(x+1)$
- $x^{100} - 4x^{98} + 5x + 6 = (x-2)(x-1)(x+1)Q(x) + R(x)$
- Try plugging in $x = 1, 2, -1$. 1 gives $R(1) = 8$, 2 gives $R(2) = 16$, and -1 gives $R(-1) = -2$
- $R(x) = ax^2 + bx + c$. We get $a + b + c = 8$, $4a + 2b + c = 16$, $a - b + c = -2$
- Now, we just solve the equations. We get $(a, b, c) = (1, 5, 2)$, so our answer is $x^2 + 5x + 2$

Remainder Theorem Problems

2016 PUMaC Algebra A # 4

Suppose that P is a polynomial with integer coefficients such that $P(1) = 2, P(2) = 3$ and $P(3) = 2016$. If N is the smallest possible positive value of $P(2016)$, find the remainder when N is divided by 2016.

- Try dividing by $(x-1)(x-2)$. $P(x) = (x-1)(x-2)Q(x) + R(x)$. Taking $x = 1, 2$, we get $R(1) = 2$, $R(2) = 3$, so $R(x) = x + 1$
- Now, plug in 3. We get $2Q(3) + 4 = 2016$. So, $Q(3) = 1006$
- Now, $P(2016) = 2015 * 2014Q(2016) + 2017$.
- Apply the lemma. We know $Q(3) = 1006$, so $2013 | Q(2016) - Q(3) = Q(2016) - 1006$
- $Q(2016) \equiv 1006 \pmod{2013}$, so we want $Q(2016) = 1006$, which gives $P(2016) = 2015 * 2014 * 1006 + 2017$
- Finally, just find the remainder when divided by 2016. This is $(-1)(-2)(1006) + 1 \equiv 2013 \pmod{2016}$

Lemma for Problem 4.2

Lemma

If $P(x)$ is an integer polynomial, then for any integers $a \neq b$, we have

$$a - b | P(a) - P(b)$$

Proof: If $P(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_0$, then

$$P(a) - P(b) = c_n(a^n - b^n) + c_{n-1}(a^{n-1} - b^{n-1}) + \ldots + c_1(a - b)$$

Note that each term is divisible by $a - b$. Hence, factoring out, we get that $a - b | P(a) - P(b)$.

Aside on Lagrange Interpolation

2017 HMMT Algebra and Number Theory #6

A polynomial P of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find $2017P(2017)$.

- $Q(x) = x^2P(x) - 1$ has roots $x = 1, 2, \ldots, 2016$
- $x^2P(x) = (c(x-1)(x-2)\ldots(x-2016))(x-k) + 1$.
- The linear and constant coefficients on the right are zero
- Constant term on the right is $1 - 2016!ck = 0$
- On the other hand, the linear coefficient on the right is pretty nasty. We get

$$c(1*2*\ldots*2016 + k(1*2*\ldots*2015 + \ldots + 2*\ldots*2016)) + 1 = 0$$

- The coefficient of k is just $2016!\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2016}\right) = 2016!H_{2016}$

Aside on Lagrange Interpolation

2017 HMMT Algebra and Number Theory #6

A polynomial P of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find $2017P(2017)$.

- Now, we just solve.
- We get $k = -\frac{1}{H_{2016}}$ and $c = \frac{-H_{2016}}{2016!}$
- Now, just find $2017P(2017)$. Our answer should be $-H_{2016}$

Quadratics

2020 AIME I #14

Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a + b)^2$.

- $P(x) = (x - r)(x - s)$. Note that the roots of $P(P(x))$ satisfy $P(x) = r$ or $P(x) = s$
- Consider 2 cases. Case 1: $3, 4$ are solutions to $P(x) = r$
- So, $(x - r)(x - s) - r$ has roots $3, 4$ and $(x - r)(x - s) - s$ has roots a, b
- By Vieta's, these two quadratics have the same sum of roots, so $a + b = 3 + 4 = 7$, and $(a + b)^2 = 49$

2020 AIME I #14

Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a+b)^2$.

- Case 2: $(3-r)(3-s) = r$, $(4-r)(4-s) = s$. a is a root of $P(x) = r$, b is a root of $P(x) = s$
- We have $9 - 4r - 3s + rs = 0$, $16 - 4r - 5s + rs = 0$
- Now, subtract the two equations to get $2s = 7$
- So, $s = \frac{7}{2}$. Plugging into the second equation, $r = -3$
- Now, we have what $P(x)$ is. The sum of the roots of $P(x) = r$, $P(x) = s$ are both $r + s = \frac{1}{2}$
- So, we get $a = \frac{1}{2} - 3 = -\frac{5}{2}$, $b = \frac{1}{2} - 4 = -\frac{7}{2}$. So, $a + b = -6 \implies (a+b)^2 = 36$
- Combining our two cases, our answer is $49 + 36 = 85$

Quadratics

ARML 2017 Tiebreaker # 1

Compute the least positive N such that there exists a quadratic polynomial $f(x)$ with integer coefficients satisfying

$$f(f(1)) = f(f(5)) = f(f(7)) = f(f(11)) = N.$$

- See the handout for a full solution