Lesson 1: Polynomials

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Definitions

- $P(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$
- $z$ is a root of $P(x)$ iff $P(z) = 0$
- In the above $P$, we denote $n$ as the degree of the polynomial, or $\text{deg } P$
- Fundamental Theorem of Algebra: Counted with multiplicity, a polynomial $P(x)$ has exactly $\text{deg } P$ roots, all of which are complex
- If $P$ has roots $z_1, \ldots, z_n$, then it can always be factored as
  \[ P(x) = c_n (x - z_1)(x - z_2)\ldots(x - z_n) \]
Vieta’s Formulas

- \(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = a_n(x - r_1)(x - r_2)\cdots(x - r_n)\)
- Suppose we expand the right hand side and compare the coefficients of \(x^k\) for each \(0 \leq k \leq n - 1\)
- The equations we get are called Vieta’s Formulas:

### Results

- \(r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}\)
- \(r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n = \frac{a_{n-2}}{a_n}\).
- \(\cdots\)
- \(r_1r_2 \cdots r_n = (-1)^n\frac{a_0}{a_n}\)
- The sum of all possible products of \(k\) roots is equal to \((-1)^k\frac{a_{n-k}}{a_n}\)
Vieta’s Example

Ex: \( P(x) = (x - r_1)(x - r_2)(x - r_3) \)

\[
P(x) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - r_1 r_2 r_3
\]
2008 AIME I # 7

Let $r$, $s$, and $t$ be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$ 

Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

- Expand $(r + s)^3 + (s + t)^3 + (t + r)^3$
- $r + s + t = 0$, $(r + s)^3 = -t^3$
- $-(r^3 + s^3 + t^3)$. No need to expand.
- $8r^3 + 1001r + 2008 = 0 \implies -r^3 = \frac{1001r + 2008}{8}$
- $-(r^3 + s^3 + t^3) = \frac{1001(r+s+t) + 2008*3}{8} = \frac{2008*3}{8} = 753$
- $r^3 + s^3 + t^3 = (r + s + t)(r^2 + s^2 + t^2 - rs - st - rt) + 3rst$
Let \( a \) and \( b \) be real numbers, and let \( r, s, \) and \( t \) be the roots of 
\[
f(x) = x^3 + ax^2 + bx - 1.
\]
Also, \( g(x) = x^3 + mx^2 + nx + p \) has roots \( r^2, s^2, \) and \( t^2 \). If \( g(-1) = -5 \), find the maximum possible value of \( b \).

- Express \( m, n, p \) in terms \( a, b \)
- \( p = -r^2s^2t^2 \). But, \( rst = 1 \), so \( p = -1 \)
- \( m = -r^2 - s^2 - t^2 \).
  \[
  r^2 + s^2 + t^2 = (r + s + t)^2 - 2(rs + st + rt) = a^2 - 2b
  \]
- \( b^2 = r^2s^2 + r^2t^2 + s^2t^2 + 2rst(r + s + t) \implies n = b^2 + 2a \)
- \( g = -1 + m - n - 1 = -2 + 2b - a^2 - b^2 - 2a = -5 \implies a^2 + 2a + (b^2 - 2b - 3) = 0 \)
- \( (a + 1)^2 + b^2 - 2b - 4 = 0 \implies b^2 - 2b - 4 \leq 0 \implies b \leq 1 + \sqrt{5} \)
Let $P(x)$ be a nonzero polynomial such that $(x - 1)P(x + 1) = (x + 2)P(x)$ for every real $x$, and $(P(2))^2 = P(3)$. Then $P\left(\frac{7}{2}\right) = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

- Plug in $x = -2, 1$. Plugging in $-2$ gives $P(-1) = 0$ and $1$ gives $P(1) = 0$, so $-1, 1$ are roots of $P(x)$.
- Now, try $x = -1$. We get $RHS = P(-1) = 0$, so $0$ is also a root.
- So, we can factor $P(x)$ as $P(x) = (x - 1)x(x + 1)Q(x)$
- Now, plug this form back into the original equation. We get $Q(x + 1) = Q(x)$, after canceling out $(x - 1)x(x + 1)(x + 2)$
- Now, what can $Q$ be? $Q(x) - Q(0)$ has roots $0, 1, 2, 3, \ldots$, so it must be the $0$ polynomial, and $Q(x) = c$ for some $c$
Let $P(x)$ be a nonzero polynomial such that $(x - 1)P(x + 1) = (x + 2)P(x)$ for every real $x$, and $(P(2))^2 = P(3)$. Then $P\left(\frac{7}{2}\right) = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

Now, what can $Q$ be? $Q(x) - Q(0)$ has roots 0, 1, 2, 3, ..., so it must be the 0 polynomial, and $Q(x) = c$ for some $c$.

$P(2) = 6c$, $P(3) = 24c$, so $c = 2/3$.

Now, we have $P(x) = \frac{2}{3}x(x - 1)(x + 1)$. Plugging in $7/2$, our final answer is $105/4$. 
Factorization

2014 USAMO/1

Let $a, b, c, d$ be real numbers such that $b - d \geq 5$ and all zeros $x_1, x_2, x_3,$ and $x_4$ of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

- Vieta’s formulas will work, but they get ugly fast
- If we plug in $z$ into $P(x)$, we get $(z - x_1)(z - x_2)(z - x_3)(z - x_4)$. How can we get the squares?
- $x^2 + 1$ is actually a difference of squares! $x^2 + 1 = (x - i)(x + i)$
- Now, the product becomes $(x_1 + i) \cdots (x_4 + i) \cdot (x_1 - i) \cdots (x_4 - i) = P(i)P(-i)$
- Now, plug back into the original polynomial. We get $P(i)P(-i) = (1 + ai - b - ci + d)(1 - ai - b + ci + d)$, which is $(1 - b + d + (a - c)i)(1 - b + d - (a - c)i) = (1 - b + d)^2 + (a - c)^2$
Let $a, b, c, d$ be real numbers such that $b - d \geq 5$ and all zeros $x_1, x_2, x_3,$ and $x_4$ of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

- Now, plug back into the original polynomial. We get
  
  $P(i)P(-i) = (1 + ai - b - ci + d)(1 - ai - b + ci + d),$ which is
  
  $(1 - b + d + (a - c)i)(1 - b + d - (a - c)i) = (1 - b + d)^2 + (a - c)^2$

- Now, $b - d \geq 5,$ so we get $(5 - 1)^2 = 16.$
Remainder Theorem

- Remainder Theorem: If $P$ has real coefficients, the remainder when $P(x)$ is divided by $x - c$ is $P(c)$.

Proof.

- $P(x)$ can be written as $(x - c)Q(x) + R(x)$, where $Q(x)$ is the quotient and $R(x)$ is the remainder.
- $\deg R < \deg(x - c) = 1$.
- Therefore, $R(x)$ is a constant function.
- Plugging in $x = c$ yields $P(c) = R(c)$, so the remainder is $P(c)$. 
Remainder Theorem Problems

AoPS Intermediate Algebra

Find the remainder when \( x^{100} - 4x^{98} + 5x + 6 \) is divided by \( x^3 - 2x^2 - x + 2 \).

- The cubic is factorable as \((x - 2)(x - 1)(x + 1)\)
- \( x^{100} - 4x^{98} + 5x + 6 = (x - 2)(x - 1)(x + 1)Q(x) + R(x) \)
- Try plugging in \( x = 1, 2, -1 \). 1 gives \( R(1) = 8 \), 2 gives \( R(2) = 16 \), and \(-1\) gives \( R(-1) = -2 \)
- \( R(x) = ax^2 + bx + c \). We get \( a + b + c = 8 \), \( 4a + 2b + c = 16 \), \( a - b + c = -2 \)
- Now, we just solve the equations. We get \((a, b, c) = (1, 5, 2)\), so our answer is \( x^2 + 5x + 2 \)
Suppose that $P$ is a polynomial with integer coefficients such that $P(1) = 2$, $P(2) = 3$ and $P(3) = 2016$. If $N$ is the smallest possible positive value of $P(2016)$, find the remainder when $N$ is divided by 2016.

Try dividing by $(x - 1)(x - 2)$. $P(x) = (x - 1)(x - 2)Q(x) + R(x)$. Taking $x = 1, 2$, we get $R(1) = 2, R(2) = 3$, so $R(x) = x + 1$.

Now, plug in 3. We get $2Q(3) + 4 = 2016$. So, $Q(3) = 1006$.


Apply the lemma. We know $Q(3) = 1006$, so $2013|Q(2016) - Q(3) = Q(2016) - 1006$.


Finally, just find the remainder when divided by 2016. This is $(-1)(-2)(1006) + 1 \equiv 2013 \pmod{2016}$.
Lemma for Problem 4.2

Lemma

If $P(x)$ is an integer polynomial, then for any integers $a \neq b$, we have

$$a - b | P(a) - P(b)$$

Proof: If $P(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_0$, then

$$P(a) - P(b) = c_n (a^n - b^n) + c_{n-1} (a^{n-1} - b^{n-1}) + \ldots + c_1 (a - b)$$

Note that each term is divisible by $a - b$. Hence, factoring out, we get that $a - b | P(a) - P(b)$. 
Aside on Lagrange Interpolation

2017 HMMT Algebra and Number Theory #6

A polynomial $P$ of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find $2017P(2017)$.

- $Q(x) = x^2P(x) - 1$ has roots $x = 1, 2, \ldots, 2016$
- $x^2P(x) = (c(x - 1)(x - 2) \ldots (x - 2016))(x - k) + 1$.
- The linear and constant coefficients on the right are zero
- Constant term on the right is $1 - 2016!ck = 0$
- On the other hand, the linear coefficient on the right is pretty nasty. We get

$$c(1 \times 2 \times \ldots \times 2016 + k(1 \times 2 \times \ldots \times 2015 + \ldots + 2 \times \ldots \times 2016)) + 1 = 0$$

- The coefficient of $k$ is just

$$2016! \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2016}\right) = 2016!H_{2016}$$
A polynomial $P$ of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find $2017P(2017)$.

- Now, we just solve.
- We get $k = -\frac{1}{H_{2016}}$ and $c = \frac{-H_{2016}}{2016!}$
- Now, just find $2017P(2017)$. Our answer should be $-H_{2016}$
Let \( P(x) \) be a quadratic polynomial with complex coefficients whose \( x^2 \) coefficient is 1. Suppose the equation \( P(P(x)) = 0 \) has four distinct solutions, \( x = 3, 4, a, b \). Find the sum of all possible values of \((a + b)^2\).

\[
P(x) = (x - r)(x - s).\]

Note that the roots of \( P(P(x)) \) satisfy \( P(x) = r \) or \( P(x) = s \).

Consider 2 cases. Case 1: 3, 4 are solutions to \( P(x) = r \)

So, \((x - r)(x - s) - r\) has roots 3, 4 and \((x - r)(x - s) - s\) has roots \(a, b\).

By Vieta’s, these two quadratics have the same sum of roots, so \(a + b = 3 + 4 = 7\), and \((a + b)^2 = 49\).
Let $P(x)$ be a quadratic polynomial with complex coefficients whose $x^2$ coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a + b)^2$.

Case 2: $(3 - r)(3 - s) = r, (4 - r)(4 - s) = s$. $a$ is a root of $P(x) = r$, $b$ is a root of $P(x) = s$.

We have $9 - 4r - 3s + rs = 0, 16 - 4r - 5s + rs = 0$.

Now, subtract the two equations to get $2s = 7$.

So, $s = \frac{7}{2}$. Plugging into the second equation, $r = -3$.

Now, we have what $P(x)$ is. The sum of the roots of $P(x) = r$, $P(x) = s$ are both $r + s = \frac{1}{2}$.

So, we get $a = \frac{1}{2} - 3 = -\frac{5}{2}, b = \frac{1}{2} - 4 = -\frac{7}{2}$. So, $a + b = -6 \implies (a + b)^2 = 36$.

Combining our two cases, our answer is $49 + 36 = 85$. 
ARML 2017 Tiebreaker # 1

Compute the least positive $N$ such that there exists a quadratic polynomial $f(x)$ with integer coefficients satisfying

$$f(f(1)) = f(f(5)) = f(f(7)) = f(f(11)) = N.$$ 

See the handout for a full solution.