Lesson 15: Number Theoretic Functions B

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General Functions

- Often, we are presented with a newly defined number theoretic function and asked to work with it
- On such problems, we have no prior knowledge about the function, unlike say the totient function
- Instead, it's important to understand *how* the function behaves to supplant this lack of prior knowledge
- In the examples, we'll discover a lot about our function before we extract the final answer

General Functions

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For any integer $k \geq 1$, let $p(k)$ be the smallest prime which does not divide k. Define the integer function $X(k)$ to be the product of all primes less than $p(k)$ if $p(k) > 2$, and $X(k) = 1$ if $p(k) = 2$. Let $\{x_n\}$ be the sequence defined by $x_0 = 1$, and $x_{n+1}X(x_n) = x_n p(x_n)$ for $n \geq 0$. Find the smallest positive integer, t such that $x_t = 2090$.

- We have the recursion $x_{n+1} = \frac{x_n p(x_n)}{X(x_n)}$
- How to interpret this?
- Given x_n, multiply by the smallest prime not dividing x_n and divide by all smaller primes
- So considering the prime factorization of x_n will be useful
- Let's write out the first few x_i:

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- $x_0 = 1$, $x_1 = 2$
- $x_2 = 3$
- $x_3 = 6 = 2 \cdot 3$
- $x_4 = 5$
- $x_5 = 10 = 2 \cdot 5$
- $x_6 = 15 = 3 \cdot 5$
- $x_7 = 30 = 2 \cdot 3 \cdot 5$
- $x_8 = 7$
- Considering the prime factorization, we find the smallest prime not in it, add it, and delete all smaller primes.
- In particular, this looks a lot like binary!
- We can rewrite the first few x_i to see this more clearly

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- $x_1 = 7^0 \cdot 5^0 \cdot 3^0 \cdot 2^1$
- $x_2 = 7^0 \cdot 5^0 \cdot 3^1 \cdot 2^0$
- $x_3 = 7^0 \cdot 5^0 \cdot 3^1 \cdot 2^1$
- $x_4 = 7^0 \cdot 5^1 \cdot 3^0 \cdot 2^0$
- $x_5 = 7^0 \cdot 5^1 \cdot 3^0 \cdot 2^1$
- $x_6 = 7^0 \cdot 5^1 \cdot 3^1 \cdot 2^0$
- $x_7 = 7^0 \cdot 5^1 \cdot 3^1 \cdot 2^1$
- $x_8 = 7^1 \cdot 5^0 \cdot 3^0 \cdot 2^0$
- So if $\overline{e_k e_{k-1} \cdots e_0}$ is the binary representation of n then $x_n = p_k^{e_k} p_{k-1}^{e_{k-1}} \cdots p_0^{e_0}$ (where $p_1 < p_2 < \cdots$ is the sequence of primes)
- We can rigorously prove this, but it is just a formalization of this intuition
- To get x_{n+1} we find the first zero in the above, flip it to a 1, and change everything after it to a 0, equivalent to adding 1 in binary

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- Now we can extract the answer
- $x_t = 2090 = 2 \cdot 5 \cdot 11 \cdot 19$
- $19^1 \cdot 17^0 \cdot 13^0 \cdot 11^1 \cdot 7^0 \cdot 5^1 \cdot 3^0 \cdot 2^1$
- $10010101_2 = \boxed{149}$

General Functions

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For positive integers n and k, let $f(n,k)$ be the remainder when n is divided by k, and for $n > 1$ let $F(n) = \max_{1 \le k \le \frac{n}{2}} f(n,k)$. Find the remainder when $\sum_{n=20}^{100} F(n)$ is divided by 1000.

- Our main task is to understand the function F for $20 \le n \le 100$
- Given n, which k should we choose to maximize $f(n,k)$?
- We want the largest multiple of k less than n to be far from n
- Equivalently, we want the smallest multiple of k more than n to be close to n
- We should choose a k such that $ki > n$ but $(k-1)i \le n$ for some integer i; otherwise $k+1$ would be a better choice

2013 AIME II # 14

- This works out to $k-1 \leq \frac{n}{i} < k$, or $k = \left\lfloor \frac{n}{i} \right\rfloor + 1$
- Since $1 \leq k \leq \frac{n}{2}$ we need $i \geq 3$
- Then
  $f(n,k) = n-(i-1)k = n-(i-1)(\left\lfloor \frac{n}{i} \right\rfloor + 1) = \left\lfloor \frac{n}{i} \right\rfloor - (i-1-n \bmod i)$
- Now, note that this is roughly $\frac{n}{i}$, so minimizing i will, in general, give a larger value of $f(n,k)$
- Because we have $n \geq 20$, the difference between $\frac{n}{3}$ and $\frac{n}{4}$ is greater than the other differences, so it will always be better to take $i = 3$.
- Now, we just need to sum $f(n, \left\lfloor \frac{n}{3} \right\rfloor + 1) = \left\lfloor \frac{n}{3} \right\rfloor - (2 - n \bmod 3)$ over all n.
- Computation of this gives us
  $3(\frac{7+33}{2})(33-7+1) - 27 - 27(0+1+2) = 1512$, so our answer is $\boxed{512}$.

Totient Function

- Recall that the *totient* function returns the number of integers in $\{1, 2, \ldots, n\}$ that are relatively prime to n
- We denote this as $\varphi(n)$
- If $p_1, p_2, \ldots, p_k$ are the distinct primes dividing n, then we have the following formula:
- $\varphi(n) = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) n$
- This can be proved using PIE and a little bit of factoring; try it on your own if you haven't seen it before
- The totient function has many nice properties stemming from both its definition and this formula, such as Euler's Theorem: if $\gcd(a, n) = 1$ then $a^{\varphi(n)} \equiv 1 \pmod{n}$
- Let's see some of them in the following examples

Classical

Let n be a positive integer. Prove that

$$\sum_{k\geq 1} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor = \frac{1}{2}n(n+1).$$

- Let's try induction!
- Let $f(n)$ be equal to the sum in the problem. What is $f(n) - f(n-1)$?
- $f(n) - f(n-1) = \sum_{k\geq 1} \varphi(k) \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor \right)$
- What exactly is the difference of floors? Remember $\lfloor n/k \rfloor$ counts the number of multiples of k at most n.
- It is nonzero when $k \mid n$.
- Therefore, $f(n) - f(n-1) = \sum_{k|n} \phi(k)$. What is this sum?

Totient Function

Lemma

$\sum_{k|n} \phi(k) = n$

Proof.

- Consider the fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ in reduced form.
- How many fractions have a denominator of k?
- If there was a fraction of the form $\frac{a}{k}$, then we must have $\gcd(a, k) = 1$ for it to be in simplest form. Therefore, there are $\phi(k)$ possible values of a.
- Every such fraction is included exactly once, so there are $\phi(k)$
- Since there are n fractions in total, $\sum_{k|n} \phi(k) = n$

□

- As $f(n) - f(n-1) = n$ for all n and $f(1) = 1$, we can easily show with induction that $f(n) = \frac{1}{2}n(n+1)$.

Totient Function

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Let $n \geq 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the nth smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

- Suppose the divisors of n are $d_1, d_2, \ldots, d_k$. We care about the interval $[1, d_1 + d_2 + \cdots + d_k]$.
- A single interval of size $d_1 + d_2 + \cdots + d_k$ is quite awkward.
- Note that k intervals of sizes $d_1, d_2, \ldots, d_k$ are much simpler
- How many numbers relatively prime to n are there in an interval $[m, m + d_i)$?
- This ranges over all residues mod d_i, so there are exactly $\phi(d_i)$ relatively prime to d_i, so at most $\phi(d_i)$ relatively prime to n
- Now, summing this over all intervals gives us $\sum_{i=1}^{k} \varphi(d_i) = n$.
- Thus, there are at most n values in the whole interval of size $\sigma(n)$ that are relatively prime to n.

TST 2018 #1

- Thus, the nth smallest value relatively prime to n is at least $\sigma(n)$, since there are at most n values up to that point.
- Now, we claim that equality only holds for n equal to a prime power.
- First, for prime power $n = p^\alpha$, what's its divisor sum? What's the nth smallest number relatively prime with it?
- $\sigma(n) = 1 + p + \ldots + p^\alpha = \frac{p^{\alpha+1}-1}{p-1}$. For the latter, note that there are $(p-1)m$ numbers relatively prime to p^α less than pm
- So, there are $p^\alpha - 1$ numbers rel prime to n less than $\frac{p^{\alpha+1}-p}{p-1}$, so our answer is
$$\frac{p^{\alpha+1}-p}{p-1} + 1 = \frac{p^{\alpha+1}-1}{p-1} = \sigma(n)$$

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- For n not a prime power, suppose we have distinct primes p, q such that $pq|n$.
- WLOG, assume $p < q$.
- If we were to make our first interval of size q, there would be a value in that interval relatively prime with the interval size but not relatively prime with n (in particular, p).
- Thus, equality cannot hold.