Lesson 14: Number Theoretic Functions A

Adithya B., Brian L., William W., Daniel X.

October 2020

Divisor Count

- Let's begin with the divisor counting function
- We use $\tau(n)$ to denote the number of divisors of n
- Recall that if n has prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ then

$$\tau(n) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1).$$

- This simple formula is enough to solve most problems related to the divisor counting function

Divisor Count

2004 AIME II # 8

How many positive integer divisors of 2004^{2004} are divisible by exactly 2004 positive integers?

- Let's start with the prime factorization:
- $2004^{2004} = 2^{4008} \cdot 3^{2004} \cdot 167^{2004}$.
- We can represent each divisor of 2004 as $2^a \cdot 3^b \cdot 167^c$ where $0 \le a \le 4008$, $0 \le b \le 2004$, $0 \le c \le 2004$
- The number of divisors of this number is $(a+1)(b+1)(c+1)$
- We now have $(a+1)(b+1)(c+1) = 2004 = 2^2 \cdot 3 \cdot 167$
- This immediately implies that a, b, c cannot be greater than 2004
- So we just need to find the number of nonnegative (a, b, c) that work

2004 AIME II # 8

- $(a+1)(b+1)(c+1) = 2^2 \cdot 3 \cdot 167$
- There are 3 ways to assign the factor of 167
- There are 3 ways to assign the factor of 3
- There are 6 ways to assign the two factors of 2:
- 3 cases in which both factors go to the same variable, 3 cases in which the factors go to different variables
- Total of $3 \cdot 3 \cdot 6 = \boxed{54}$

Divisor Count

2005 AIME I # 12

For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by

$$S(n) = \tau(1) + \tau(2) + ... + \tau(n).$$

Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.

- You might know when $\tau(n)$ is odd
- $\tau(n) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$
- This is odd if and only if each factor $e_i + 1$ is odd
- In other words, each e_i is even
- So $\tau(n)$ is odd if and only if n is a square.

2005 AIME I # 12

- So $S(1), S(2), S(3)$ are odd (they are the sum of the odd $\tau(1)$ and some even numbers)
- $S(4), S(5), S(6), S(7), S(8)$ are even (they are the sum of odds $\tau(1), \tau(4)$ and some even numbers)
- $S(9), S(10), \ldots, S(15)$ are odd, and so on
- Note that $1936 = 44^2 < 2005 < 45^2$
- $S(n)$ is odd for $n \in [1^2, 2^2 - 1] \cup [3^2, 4^2 - 1] \cup \cdots \cup [43^2, 44^2 - 1]$
- $S(n)$ is even for $n \in [2^2, 3^2 - 1] \cup [4^2, 5^2 - 1] \cup \cdots \cup [44^2, 2005]$
- $a = 44^2 - 43^2 + 42^2 - 41^2 + \cdots + 2^2 - 1^2$
- $a = 87 + 83 + 79 + \cdots + 3$
- $b = 2006 - 44^2 + 43^2 - 42^2 + \cdots + 3^2 - 2^2$
- $b = 70 + 85 + 81 + \cdots + 5$
- $b - a = (70 - 87) + (85 - 83) + (81 - 79) + \cdots + (5 - 3)$
- $b - a = -17 + 2 \cdot 21 = \boxed{25}$

Divisor Count

2016 AIME II # 11

For positive integers N and k, define N to be k-nice if there exists a positive integer a such that a^k has exactly N positive divisors. Find the number of positive integers less than 1000 that are neither 7-nice nor 8-nice.

- Let $a = p_1^{e_1} p_2^{e_2} \cdots p_i^{e_i}$.
- What condition is there on $N = \tau(a^k)$ where $\tau(n)$ denotes the number of divisors of n?
- Since $a^k = p_1^{ke_1} p_2^{ke_2} \cdots p_i^{ke_i}$, $N = (ke_1 + 1)(ke_2 + 1) \cdots (ke_i + 1)$.
- This means we must have $N \equiv 1 \pmod{k}$. Is this condition sufficient?
- Yes, choose $a = 2^{\frac{N-1}{k}}$ so that $a^k = 2^{N-1}$, which has N divisors.
- So, we need to find the number of integers that are neither 1 mod 7 nor 1 mod 8.

- We can use the principle of inclusion and exclusion.
- There are $\left\lceil \frac{999}{7} \right\rceil = 143$ integers that are 1 mod 7.
- There are $\left\lceil \frac{999}{8} \right\rceil = 125$ integers that are 1 mod 8.
- There are $\left\lceil \frac{999}{56} \right\rceil = 18$ integers that are 1 mod 56.
- By PIE, the answer is $999 - 143 - 125 + 18 = \boxed{749}$.

Multiplicative Functions

- One useful property that some number theoretic functions have is *multiplicativity*
- The definition is slightly different from the definition for real numbers
- $f : \mathbb{N} \mapsto \mathbb{N}$ is *multiplicative* if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime
- This slightly more general definition encompasses many more functions in number theory
- For example, the totient function is multiplicative: check that $\varphi(mn) = \varphi(m)\varphi(n)$ whenever $\gcd(m, n) = 1$
- In order to understand a multiplicative function, we only need to consider where it sends prime powers: using multiplicativity we can then describe the entire function

Multiplicative Functions

2016 PUMaC NT #7

Compute the number of positive integers between 2017 and 2017^2 such that $n^n \equiv 1 \pmod{2017}$.

- How can we simplify $n^n \pmod{2017}$?
- Fermat's Little Theorem!
- Let $x \equiv n \pmod{2017}$ and let $y \equiv n \pmod{2016}$. Note that the pair (x, y) uniquely determines n as the range covers $2017 \cdot 2016$ consecutive integers.
- $1 \equiv n^n \equiv x^n \equiv x^y \pmod{2017}$.
- How can we count the pairs (x, y)? How can we reduce the problem?
- Use primitive roots! Let $x = g^k$ where g is a primitive root (primitive root exists since 2017 is a prime)
- $x^y \equiv g^{ky} \equiv 1 \pmod{2017}$
- As g is a primitive root, $2016 \mid ky$. This implies $\frac{2016}{\gcd(2016,k)} \mid y$.

2016 PUMaC NT #7

- y has 2016 possible values, so the number of y that satisfy the above divisibility is $\frac{2016}{\frac{2016}{\gcd(2016,k)}} = \gcd(2016, k)$.
- As k ranges from 1 to 2016, the number of pairs is $\sum_{k=1}^{2016} \gcd(k, 2016)$.
- Let $f(n) = \sum_{k=1}^{n} \gcd(k, n)$. How can we evaluate this?
- Let's show f is multiplicative. That is, for relatively prime a, b, we will prove $f(a)f(b) = f(ab)$.

2016 PUMaC NT #7

- y has 2016 possible values, so the number of y that satisfy the above divisibility is $\frac{2016}{\frac{2016}{\gcd(2016,k)}} = \gcd(2016, k)$.
- As k ranges from 1 to 2016, the number of pairs is $\sum_{k=1}^{2016} \gcd(k, 2016)$.
- Let $f(n) = \sum_{k=1}^{n} \gcd(k, n)$. How can we evaluate this?
- Let's show f is multiplicative. That is, for relatively prime a, b, we will prove $f(a)f(b) = f(ab)$.

$$f(a)f(b) = \left(\sum_{i=1}^{a} \gcd(i, a)\right)\left(\sum_{j=1}^{b} \gcd(j, b)\right) = \sum_{i=1}^{a}\sum_{j=1}^{b} \gcd(i, a)\gcd(j, b).$$

2016 PUMaC NT #7

- y has 2016 possible values, so the number of y that satisfy the above divisibility is $\frac{2016}{\frac{2016}{\gcd(2016,k)}} = \gcd(2016, k)$.
- As k ranges from 1 to 2016, the number of pairs is $\sum_{k=1}^{2016} \gcd(k, 2016)$.
- Let $f(n) = \sum_{k=1}^{n} \gcd(k, n)$. How can we evaluate this?
- Let's show f is multiplicative. That is, for relatively prime a, b, we will prove $f(a)f(b) = f(ab)$.

$$f(a)f(b) = \left(\sum_{i=1}^{a} \gcd(i, a)\right)\left(\sum_{j=1}^{b} \gcd(j, b)\right) = \sum_{i=1}^{a}\sum_{j=1}^{b} \gcd(i, a)\gcd(j, b).$$

- Let k be the unique integer such that $k \equiv i \pmod{a}$ and $k \equiv j \pmod{b}$. Note that $\gcd(i, a) = \gcd(k, a)$ and $\gcd(j, b) = \gcd(k, b)$

2016 PUMaC NT #7

- $f(a)f(b) = \sum_{k=1}^{ab} \gcd(k, a) \gcd(k, b)$.
- Since a, b are relatively prime $\gcd(k, a) \gcd(k, b) = \gcd(k, ab)$.
- Thus, $f(a)f(b) = \sum_{k=1}^{ab} \gcd(k, ab) = f(ab)$, so f is multiplicative.
- We want to find $f(2016) = f(32)f(9)f(7)$.
- $f(32) = \sum_{k=1}^{32} \gcd(k, 32) = 16 \cdot 1 + 8 \cdot 2 + 4 \cdot 4 + 2 \cdot 8 + 1 \cdot 16 + 1 \cdot 32 = 112$
- $f(9) = \sum_{k=1}^{9} \gcd(k, 9) = 6 \cdot 1 + 2 \cdot 3 + 1 \cdot 9 = 21$
- $f(7) = \sum_{k=1}^{9} \gcd(k, 7) = 6 \cdot 1 + 1 \cdot 7 = 13$
- Therefore, $f(2016) = 112 \cdot 21 \cdot 13 = \boxed{30576}$.

Multiplicative Functions

2019 HMMT Algebra and Number Theory #8

There is a unique function $f : \mathbb{N} \to \mathbb{R}$ such that $f(1) > 0$ and such that

$$\sum_{d|n} f(d) f\left(\frac{n}{d}\right) = 1$$

for all $n \geq 1$. What is $f(2018^{2019})$?

- We begin by letting $n = 1$, so that $f(1) = 1$.
- We claim that f is multiplicative, that $f(ab) = f(a)f(b)$ for relatively prime a, b.
- Note that every divisor of ab can be written uniquely as the product $d_1 d_2$, where $d_1 | a$ and $d_2 | b$.
- Now, we wish to show our claim by induction on ab. Suppose it holds true for all smaller ab.

2019 HMMT Algebra and Number Theory #8

- Now, plugging in $n = a, b, ab$ gives us $1 = \sum_{d_1|a} f(d_1) f\left(\frac{a}{d_1}\right) \sum_{d_2|b} f(d_2) f\left(\frac{b}{d_2}\right) = \sum_{d_1 d_2|ab} f(d_1 d_2) f\left(\frac{ab}{d_1 d_2}\right) - 2f(1)f(ab) + 2f(a)f(b) = \sum_{d_1 d_2|ab} f(d_1 d_2) f\left(\frac{ab}{d_1 d_2}\right)$.
- Thus, we find $f(ab) = \frac{f(a)f(b)}{f(1)} = f(a)f(b)$, as desired.
- Now, note that $f(p^k)$ is independent on the prime p, which can be shown by induction on k.
- Let $a_k = f(p^k)$
- Now, consider the generating function $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Note that it satisfies $g(x)^2 = 1 + x + x^2 + \cdots = \frac{1}{1-x}$.
- Thus, we find that $g(x) = (1-x)^{-\frac{1}{2}}$.
- By the extended binomial theorem, we note that this gives us $a_n = \binom{-\frac{1}{2}}{n} = \frac{(-1)^n \binom{2n}{n}}{4^n}$.

2019 HMMT Algebra and Number Theory #8

- Note that the prime factorization of $2018^{2019} = 2^{2019} \cdot 1009^{2019}$, so the answer we seek is $a_{2019}^2 = \boxed{\frac{\binom{4038}{2019}^2}{4^{4038}}}$.