Lesson 9: Geometry B

Adithya B., Brian L., William W., Daniel X.

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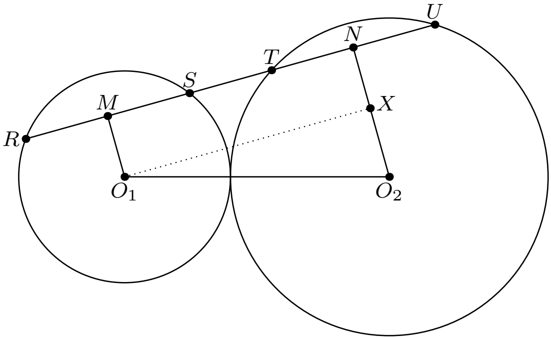
Problem of the Week

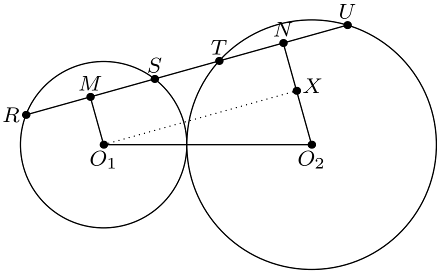
PotW

Circles Ω_1 and Ω_2 are externally tangent and have radii 10 and 15, respectively. A line intersects Ω_1 at R and S and intersects Ω_2 at T and U, in that order. If $RS = TU = 2ST$, then ST^2 can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m, n. Find the value of $m + n$.



Problem of the Week

- Let M and N be the midpoints of RS and TU, respectively. Then, $RM = MS = ST = TN = NU = x$.
- Note that $O_1M \perp RS$ and $O_2N \perp TU$.
- From the Pythagorean Theorem, $O_1M = \sqrt{100 - x^2}$ and $O_2N = \sqrt{225 - x^2}$.
- Let X be the foot of the perpendicular from O_1 to O_2N.
- Then, $O_1X = 3x$, $O_2X = \sqrt{225 - x^2} - \sqrt{100 - x^2}$, and $O_1O_2 = 25$. By the Pythagorean Theorem, $(3x)^2 + (\sqrt{225 - x^2} - \sqrt{100 - x^2})^2 = 25^2$.



Problem of the Week

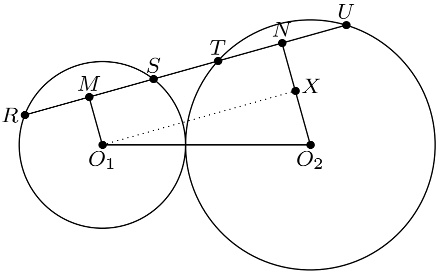
- Expanding, $7x^2 + 325 - 2\sqrt{(225 - x^2)(100 - x^2)} = 625$, or

$$7x^2 - 300 = 2\sqrt{(225 - x^2)(100 - x^2)}$$

- Squaring, $49x^4 - 4200x^2 + 90000 = 4x^4 - 1300^2 + 90000$, so $45x^4 = 2900x^2$.
- Therefore, $x^2 = \frac{580}{9} \implies \boxed{589}$.

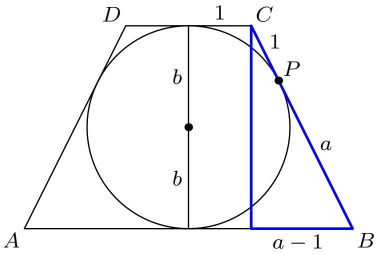
2014 AMC 12B #19

A sphere is inscribed in a truncated right circular cone as shown. The volume of the truncated cone is twice that of the sphere. What is the ratio of the radius of the bottom base of the truncated cone to the radius of the top base of the truncated cone?



2014 AMC 12B #19

- Let's assume the top base has radius 1, the bottom base has radius a, and the radius of the sphere is b.
- Note $BP = a$ and $CP = 1$. Thus, Pythagorean on the blue yields $(2b)^2 + (a-1)^2 = (a+1)^2$ or $a = b^2$.
- To find the volume of the frustum, we can find the volume of the whole cone and subtract the truncated portion.
- If h is the height of the small cone, and H is the height of the large cone, then $H - h = 2b$, and $\frac{h}{H} = \frac{1}{a}$.
- Solving $h = \frac{2b}{a-1}$ and $H = \frac{2ab}{a-1}$.



2014 AMC 12B #19

- The volume of the frustum is now

$$\frac{1}{3}\pi a^2 \frac{2ab}{a-1} - \frac{1}{3}\pi \frac{2b}{a-1} = \frac{1}{3}\pi b \frac{a^3-1}{a-1} = \frac{2}{3}\pi b(b^4+b^2+1)$$

- The volume of the sphere is $\frac{4}{3}\pi b^3$. Since the frustum has twice the volume of the sphere,

$$\frac{8}{3}\pi b^3 = \frac{2}{3}\pi b(b^4+b^2+1) \implies 4b^2 = b^4+b^2+1$$

- Thus, we have $b^4 - 3b^2 + 1 = 0$ or $a^2 - 3a + 1 = 0$. Solving for a, we get $a = \frac{3+\sqrt{5}}{2}$.

3D Geometry

2015 PUMaC Geometry A #4

Find the largest r such that 4 balls each of radius r can be packed into a regular tetrahedron with side length 1. In a packing, each ball lies outside every other ball, and every ball lies inside the boundaries of the tetrahedron. If r can be expressed in the form $\frac{\sqrt{a}+b}{c}$ where a, b, c, are integers such that $\gcd(b, c) = 1$, what is $a + b + c$?

- If we connect the centers of the centers of the spheres, we get a similar, smaller tetrahedron with side length $2r$.
- To find the ratio between the two tetrahedrons, we can find the distance from the center to a face.
- Consider some regular tetrahedron $ABCD$ with side length s, and let O be its center. Let x be the distance from O to a face.
- Note that the volume of $ABCD$ can be broken down into the volumes of $OABC$, $OACD$, $OABD$, and $OBCD$, which each have base $\frac{s^2\sqrt{3}}{4}$ and height x.

2015 PUMaC Geometry A #4

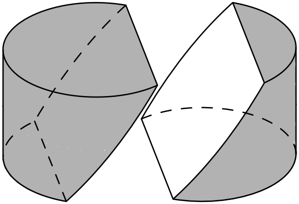
- To find the total volume of the tetrahedron, we need the height. Let G be the center of ABC.
- To find DG, we will use right triangle DGA with hypotenuse s and leg $\frac{s}{\sqrt{3}}$.
- By the Pythagorean Theorem, $DG = \frac{\sqrt{6}}{3}s$
- The total volume can be written in two ways:

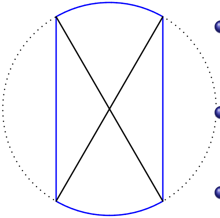
$$4 \cdot \frac{1}{3} \cdot \frac{s^2\sqrt{3}}{4} \cdot x = \frac{1}{3} \cdot \frac{s^2\sqrt{3}}{4} \cdot \frac{\sqrt{6}}{3}s$$

- We get that $x = \frac{\sqrt{6}}{12}s$.
- Returning to the problem, the distance from the center to a face in the small tetrahedron (formed by the centers of the spheres is $\frac{\sqrt{6}}{6}r$.
- For the tetrahedron with side length 1, it is equal to $\frac{\sqrt{6}}{12} = \frac{\sqrt{6}}{6}r + r$.
- Thus,$r = \frac{\frac{\sqrt{6}}{12}}{\frac{\sqrt{6}}{6}+1} = \frac{\sqrt{6}-1}{10}$. The answer is $6 - 1 + 10 = \boxed{15}$.

2015 AIME I #15

A block of wood has the shape of a right circular cylinder with radius 6 and height 8, and its entire surface has been painted blue. Points A and B are chosen on the edge on one of the circular faces of the cylinder so that arc AB on that face measures 120°. The block is then sliced in half along the plane that passes through point A, point B, and the center of the cylinder, revealing a flat, unpainted face on each half. The area of one of those unpainted faces is $a \cdot \pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a + b + c$.

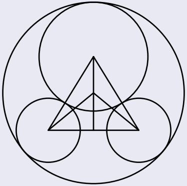


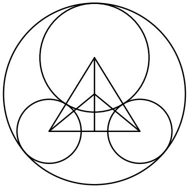


- The cross-section as is seems difficult to handle, but its projection to the base seems much more manageable.
- Now, the area of the projection is much easier to calculate, since it consists of 2 60° arcs, and 2 $30-30-120$ triangles.
- The area can be calculated to be $2\left(\frac{\pi 6^2}{6}\right)+2(6^2)\left(\frac{\sqrt{3}}{4}\right)=12\pi+18\sqrt{3}$
- Now, we just need to find how the original area relates to the area of the projection.
- Note that the two vertical lines in the projection are 6 units apart, while in the original figure, they are $\sqrt{6^2+8^2}=10$ units apart.
- Thus, we note that the area is $\frac{10}{6}$ that of the projection, so the area is $20\pi+30\sqrt{3}$, so the answer is $20+30+3=\boxed{53}$

2020 Purple Comet Math Meet #30

Four small spheres each with radius 6 are each internally tangent to a larger sphere with radius 17. The four small spheres form a ring with each of the four sphere externally tangent to its two neighboring small spheres. A sixth intermediately sized sphere is internally tangent to the large sphere and externally tangent to each of the four small spheres. Its radius is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.





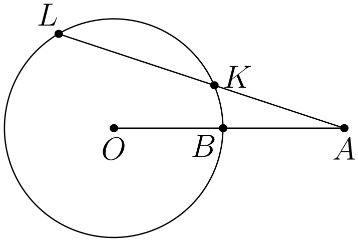
- We take a cross section through the centers of the large and intermediate spheres, as well as two opposite smaller spheres.
- Note that the two smaller spheres are opposite sides of a square of side length 12, so they are $12\sqrt{2}$ apart.
- Now, suppose the radius of the intermediate sphere is r.
- Then, we can find the equation $(6\sqrt{2})^2 + \left(\sqrt{(17-6)^2-(6\sqrt{2})^2}+(17-r)\right)^2 = (r+6)^2$
- This equation simplifies to $r^2 - 48r + 648 = r^2 + 12r + 36$, so $r = \frac{612}{60} = \frac{51}{5}$, so our answer is $51 + 5 = \boxed{56}$

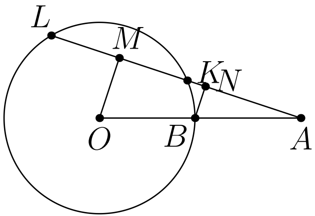
Geometric Optimization

- In some geometry problems, we are asked to maximize/minimize a certain quantity
- These problems are often the same as "normal" geometry problems, in that we still have to compute lengths, angles, areas, etc.
- However, we will have a variable x in our calculation, and will need to use algebraic methods to extract the final answer once we write the desired quantity in terms of x
- The diagrams can get quite complicated and quantities can be hard to compute, and it is important to *make geometric observations* before jumping into the computation
- A good strategy is to make geometric reductions until the method of computation becomes clear
- Also, we need to check that the maximum/minimum is achievable! In a short-answer contest a cursory check will do; in a proof be sure to justify it.

2013 AIME II # 10

Given a circle of radius $\sqrt{13}$, let A be a point at a distance $4+\sqrt{13}$ from the center O of the circle. Let B be the point on the circle nearest to point A. A line passing through the point A intersects the circle at points K and L. The maximum possible area for $\triangle BKL$ can be written in the form $\frac{a-b\sqrt{c}}{d}$, where a, b, c, and d are positive integers, a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

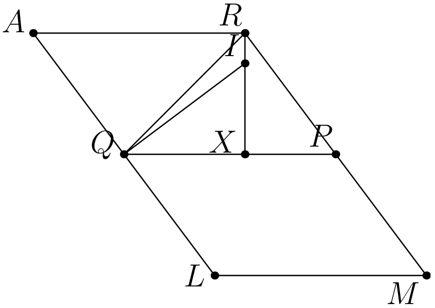


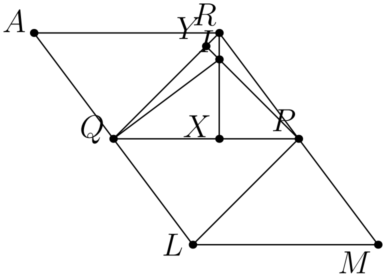


- How can we express $[BKL]$?
- $\frac{1}{2}KL \cdot BN$
- We have parallel lines OM and BN
- Then $\frac{OM}{AO} = \frac{BN}{AB}$
- We know $AO = 4 + \sqrt{13}$, $AB = 4$
- So $BN = \frac{4}{4+\sqrt{13}}OM$
- Area is now $\frac{1}{2}KL \cdot \frac{4}{4+\sqrt{13}}OM$
- This is $\frac{4}{4+\sqrt{13}}[OKL]$
- How can we maximize $[OKL]$?
- $[OKL] = \frac{1}{2}OK \cdot OL \sin \angle KOL$
- $= \frac{13}{2} \sin \angle KOL \leq \frac{13}{2}$
- $[BKL] \leq \frac{13}{2} \cdot \frac{4}{4+\sqrt{13}} = \frac{104-26\sqrt{13}}{3} \implies \boxed{146}$

2019 ARML Individual #8

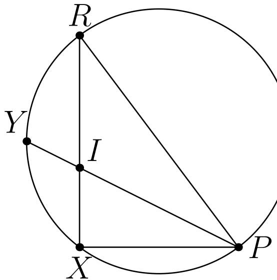
In parallelogram $ARML$, points P and Q are the midpoints of sides $\overline{RM}$ and $\overline{AL}$, respectively. Point X lies on segment $\overline{PQ}$, and $PX = 3$, $RX = 4$, and $PR = 5$. Point I lies on segment $\overline{RX}$ such that $IA = IL$. Compute the maximum possible value of $\frac{[PQR]}{[LIP]}$.





- Lots of stuff going on in the diagram
- $3-4-5$ triangle $RXP \implies RI \perp PQ$
- IQ perpendicular bisector of AL
- $AL \parallel RM \implies QI \perp RM$
- I orthocenter of $\triangle PQR$
- $PI \perp QR$
- $PRQL$ parallelogram $\implies PI \perp LP$
- Now we can handle $[LIP]$!
- $[LIP] = \frac{1}{2}LP \cdot PI = \frac{1}{2}QR \cdot PI$
- Let PI meet QR at Y
- $[PQR] = \frac{1}{2}QR \cdot PY$
- $\frac{[PQR]}{[LIP]} = \frac{PY}{PI}$
- $\angle PXR = \angle PYR = 90^\circ \implies RPXY$ cyclic

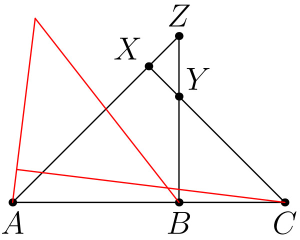
2019 ARML Individual #8



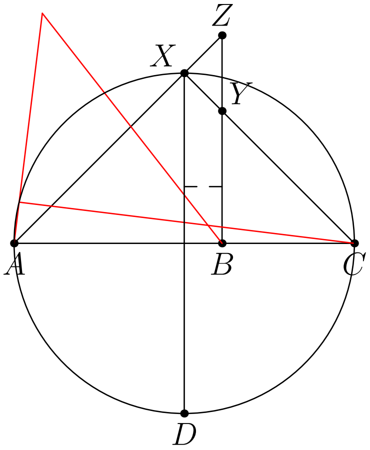
- $\angle PXR = \angle PYR = 90^\circ \implies RPXY$ cyclic
- $I \in XR$, maximize $\frac{PY}{PI}$
- Use Ratio Lemma on $\triangle XIY$
- $\frac{PY}{PI} = \frac{XY}{XI} \cdot \frac{\sin \angle YXP}{\sin \angle IXP}$
- $XY = 5 \sin \angle XPY$
- $XI = 3 \frac{\sin \angle XPY}{\cos \angle XPY}$
- $\angle YXP = 90^\circ + \angle YXR = 90^\circ + \angle RPY$
- $\sin \angle YXP = \cos \angle RPY, \sin \angle IXP = 1$
- $\frac{PY}{PI} = \frac{5}{3} \cos \angle XPY \cos \angle RPY$
- $= \frac{5}{6}(\cos \angle XPR + \cos(\angle XPY - \cos \angle RPY))$
- Maximized at $\angle XPY = \angle RPY$
- $\frac{5}{6}(\cos \angle XPR + 1) = \boxed{4/3}$

2016 MPfG #19

In the coordinate plane, consider points $A = (0, 0)$, $B = (11, 0)$, and $C = (18, 0)$. Line ℓ_A has slope 1 and passes through A. Line ℓ_B is vertical and passes through B. Line ℓ_C has slope -1 and passes through C. The three lines ℓ_A, ℓ_B, ℓ_C begin rotating clockwise about points A, B, and C, respectively. They rotate at the same angular rate. At any given time, the three lines form a triangle. Determine the largest possible area of such a triangle.

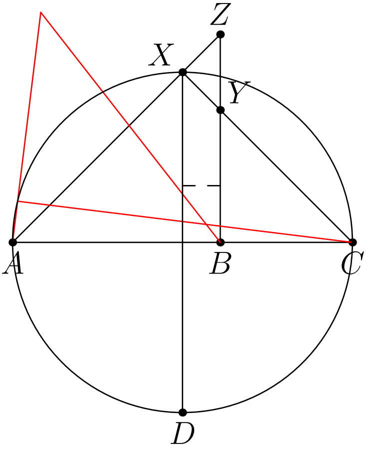


2016 MPfG #19



- Can we say anything nice about the triangle formed by ℓ_A, ℓ_B, ℓ_C?
- It's always a 45-45-90 so it suffices to maximize one of its lengths
- Note that $\angle AXC$ is always 90, so it lies on a circle with diameter AC
- If we let D be the midpoint of arc AC, what is $\angle DXC$?
- $\angle DXC = 45 = \angle XYZ$, so $DX \parallel BY$
- What is minimum distance between lines DX and BZ?

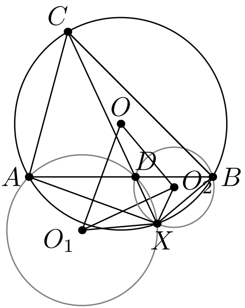
2016 MPfG #19

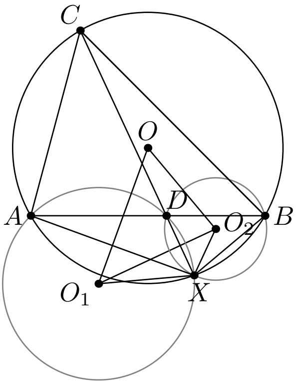


- Since BD is always a transversal, the distance between DX and BZ is always at least BD. What does this distance represent?
- BD is the maximum height of XYZ and can be computed with Pythagorean theorem as $\sqrt{2^2 + 9^2} = \sqrt{85}$
- So, the maximum area of XYZ is $\sqrt{85}^2 = 85$. Is this achievable?
- Yes! X ranges over the entire circle, so we need $BD \perp DX$, which will happen eventually. So, our answer is $\boxed{85}$.

USOMO 2020/1

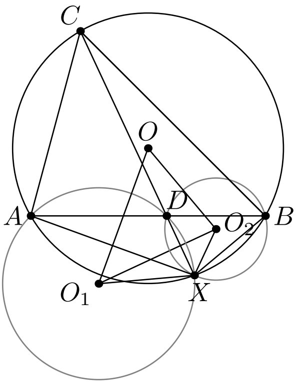
Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.





- Let's start similarly to the above question: What can we say about triangle OO_1O_2?
- The line connecting the centers is perpendicular to the radical axis, so O_1O is a 90 degree rotation of AX and O_1O_2 is a 90 degree rotation of CX
- Hence, $\angle OO_1O_2 = \angle AXC = \angle B$
- Similarly, $\angle O_1O_2O = \angle C$, so $OO_1O_2 \sim CBA$
- It suffices to minimize the length of O_1O_2. Can we find a lower bound for it?

USOMO 2020/1



- O_1O_2 is at least its projection onto AB. What is this projection's length?
- O_1's projection is the midpoint of AD and O_2's is that of BD. So, the length of its projection is always $\frac{AB}{2}$
- Hence, $O_1O_2 \geq \frac{AB}{2}$, so $[OO_1O_2] \geq \frac{[ABC]}{4}$. Can equality be achieved?
- Equality is achieved when $O_1O_2 \parallel AB$. As $O_1O_2 \perp CX$, this means $CX \perp AB$