Lesson 8: Geometry A

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Problem of the Week

PotW

The sum

$$\sum_{k=0}^{2018}\left(\cos\left(\frac{\pi k}{2019}\right)\right)^{2020}$$

can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Determine the remainder when m is divided by 1009.

- Seems hard to evaluate directly with trigonometry
- Let's use complex numbers! Let $\omega = e^{\frac{i\pi}{2019}}$ (a 4038th root of unity)
- What is $\omega + \frac{1}{\omega}$?
- Similarly, $\cos\left(\frac{\pi k}{2019}\right) = \frac{1}{2}\left(\omega^k + \frac{1}{\omega^k}\right)$.
- The sum can be rewritten as

$$\frac{1}{2^{2020}}\sum_{k=0}^{2018}\left(\omega^k + \frac{1}{\omega^k}\right)^{2020} = \frac{1}{2^{2020}}\sum_{k=0}^{2018}\sum_{i=0}^{2020}\binom{2020}{i}\omega^{k(2020-i)}\frac{1}{\omega^{ki}}.$$

Problem of the Week

- We can simplify and swap the order of the summation:

$$\frac{1}{2^{2020}}\sum_{k=0}^{2018}\sum_{i=0}^{2020}\binom{2020}{i}\omega^{k(2020-2i)} = \frac{1}{2^{2020}}\sum_{i=0}^{2020}\binom{2020}{i}\sum_{k=0}^{2018}\omega^{k(2020-2i)}.$$

- Note that $\omega^{2020-2i} = (\omega^2)^{1010-i}$, and ω^2 is a 2019th root of unity.
- Let $z = (\omega^2)^{1010-i}$. Then, $\sum_{k=0}^{2018} z^k = \frac{z^{2019}-1}{z-1} = 0$ when $z \neq 1$.
- Every term cancels except when $i = 1010$. Hence, the inner sum is just equal to 2019, and the total sum is

$$\frac{1}{2^{2020}}\binom{2020}{1010}\cdot 2019.$$

- The number of powers of 2 in $\binom{2020}{1010}$ is

$$\nu_2(2020!) - 2\nu_2(1010!) = 2013 - 2\cdot 1003 = 7.$$

Problem of the Week

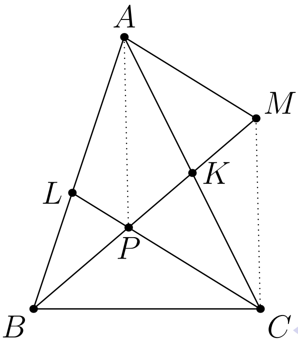
- The numerator of the result is $\frac{2019\binom{2020}{1010}}{2^7}$.
- Clearly $2019 \equiv 1 \pmod{1009}$. To find $\binom{2020}{1010} \pmod{1009}$, we can either use Lucas's theorem, or we can use the following:

$$\begin{aligned}
\frac{2020!}{(1010!)^2} &= \frac{1008! \cdot 1009 \cdot (1010 \cdot 1011 \cdots 2017) \cdot (2 \cdot 1009) \cdot 2019 \cdot 2020}{(1008!)^2 \cdot 1009^2} \\
&\equiv \frac{1008! \cdot (1 \cdot 2 \cdots 1008) \cdot 2 \cdot 2019 \cdot 2020}{(1008!)^2} \\
&\equiv \frac{(1008!)^2 \cdot 2 \cdot 1 \cdot 2}{(1008!)^2} \\
&\equiv 4 \pmod{1009}
\end{aligned}$$

- So now, we want to find $4/2^7 = 32^{-1} \pmod{1009}$, which we can calculate to be $\boxed{473}$.

2009 AIME I #5

Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.



2009 AIME I #5

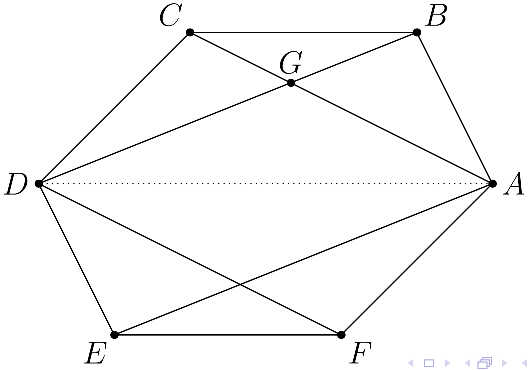
- Note that $AK = CK$, $PK = MK$, and $\angle AKM = \angle CKP$, so we see $\triangle AKM \cong \triangle CKP$.
- In particular, we see that $AMCP$ is a parallelogram and $CL \parallel AM$.
- By the angle bisector theorem, $\frac{AL}{LB} = \frac{AC}{BC} = \frac{3}{2}$.
- Also, since $LP \parallel AM$, $\triangle BLP \sim \triangle BAM$, we have

$$\frac{LP}{AM} = \frac{BL}{BA} = \frac{1}{1 + \frac{LA}{LB}} = \frac{2}{5}$$

- Thus, $LP = \frac{2}{5}AM = \boxed{072}$.

2018 HMMT Geometry # 4

Convex hexagon $ABCDEF$ is drawn in the plane such that $ACDF$ and $ABDE$ are parallelograms with area 168. AC and BD intersect at G. Given that the area of AGB is 10 more than the area of CGB, find the smallest possible area of hexagon $ABCDEF$.



2018 HMMT Geometry # 4

- $[ACDF] = [ABDE] \implies [ABD] = [ACD]$, so B, C are the same distance from AD.
- Therefore, $AD \parallel BC$.
- Let $x = [CGB]$ and $x + 10 = [AGB]$.
- $[DGA] = \frac{DG}{GB}[AGB] = \frac{AG}{GC}(x+10) = \frac{(x+10)^2}{x}$.
- $[BAD] = [BGA] + [DGA] = (x+10) + \frac{(x+10)^2}{x} = \frac{1}{2}(168) = 84$.
- $x^2 - 27x + 50 = 0$
- $(x-2)(x-25) = 0$, so $x = 2, 25$.
- $[ABCD] = [ACD] + [ABC] = \frac{1}{2}(168) + x + (x+10)$
- Let $x = 2$, so this is at least $84 + 2 + 12 = 98$.
- $ABCD$ and $DEFA$ are symmetric about the midpoint of AD, so $[ABCD] = [DEFA]$.
- Thus, we have $[ABCDEF] = 2[ABCD] = 2(98) = \boxed{196}$.

Trigonometry

- Recall that in a right triangle ABC with hypotenuse BC, the *sine* and *cosine* of $\angle B$ are defind as

$$\sin \angle B = \frac{AC}{BC}, \ \cos \angle B = \frac{AB}{BC}$$

- The unit circle can be used to extend the definitions of these trig functions to all angles
- The two most important and useful geometric facts about trig functions are the *law of sines*:

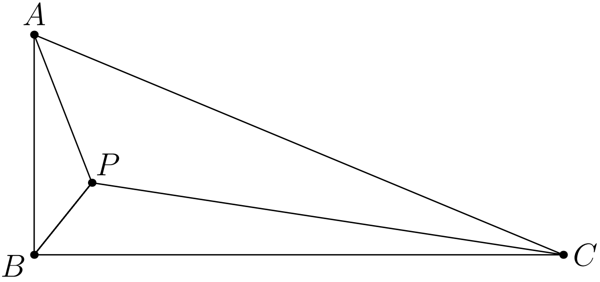
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

- and the *law of cosines*:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

1987 AIME # 9

Triangle ABC has right angle at B, and contains a point P for which $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.

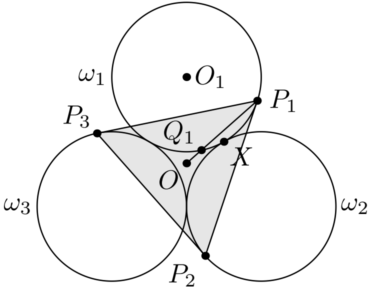


1987 AIME # 9

- We immediately know that $\angle APB = \angle BPC = \angle CPA = 120^\circ$
- Given $AP = 10$, $BP = 6$, and $\angle APB = 120^\circ$, we can use the Law of Cosines to find AB: $AB^2 = 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cos 120^\circ = 196$
- Unfortunately, this is all we can compute with the given lengths
- Let $PC = x$ be a variable; let's find more lengths in terms of x
- Law of Cosines on $\triangle PBC$:
- $BC^2 = 6^2 + x^2 - 2 \cdot 6x \cos 120^\circ = x^2 + 6x + 36$
- Law of Cosines on $\triangle PAC$:
- $AC^2 = 10^2 + x^2 - 2 \cdot 10x \cos 120^\circ = x^2 + 10x + 100$
- We now AB, BC, CA and that $\triangle ABC$ is right, so we can write another equation using the Pythagorean Theorem!
- $196 + (x^2 + 6x + 36) = (x^2 + 10x + 100)$
- Solve to get $x = \boxed{33}$

2018 AMC 12B # 25

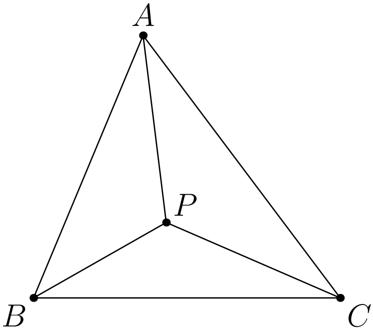
Circles ω_1, ω_2, and ω_3 each have radius 4 and are placed in the plane so that each circle is externally tangent to the other two. Points P_1, P_2, and P_3 lie on ω_1, ω_2, and ω_3 respectively such that $P_1P_2 = P_2P_3 = P_3P_1$ and line P_iP_{i+1} is tangent to ω_i for each $i = 1, 2, 3$, where $P_4 = P_1$. The area of $\triangle P_1P_2P_3$ can be written in the form $\sqrt{a} + \sqrt{b}$ for positive integers a and b. What is $a + b$?



- $\angle OP_1P_2 = 30^\circ$.
- Suppose OP_1 intersects ω_1 at some point Q_1 such that $P_1 \neq Q_1$.
- $\angle OP_1P_2 = \frac{1}{2}\angle Q_1O_1P_1$, so $\angle Q_1O_1P_1 = 60^\circ$. Thus, $P_1Q_1 = 4$.
- Length of the tangent from O to ω_1 is $\frac{4}{\sqrt{3}}$ because O_1OX is a 30-60-90 triangle.
- Power of O with respect to ω_1 is $\left(\frac{4}{\sqrt{3}}\right)^2 = \frac{16}{3}$.
- By Power of a Point, we also have $OP_1 \cdot OQ_1 = \frac{16}{3}$.
- Let $x = OP_1$, so $x(x-4) = \frac{16}{3}$.
- $x = 2 + 2\sqrt{\frac{7}{3}}$.
- $[P_1P_2P_3] = 3[P_1OP_2] = 3\left(\frac{1}{2}OP_1 \cdot OP_2 \sin 120^\circ\right) = \frac{3\sqrt{3}}{4}OP_1^2$
- $\frac{3\sqrt{3}}{4}OP_1^2 = 10\sqrt{3} + 6\sqrt{7} = \sqrt{300} + \sqrt{252}$, so our answer is $300 + 252 = \boxed{552}$.

1999 AIME I # 14

Point P is located inside traingle ABC so that angles PAB, PBC, and PCA are all congruent. The sides of the triangle have lengths $AB = 13, BC = 14$, and $CA = 15$, and the tangent of angle PAB is m/n, where m and n are relatively prime positive integers. Find $m + n$.



1999 AIME I # 14

- Let $\theta = \angle PAB = \angle PBC = \angle PCA$.
- To try to relate a trigonometry function of θ to the sides, we can use the Law of Cosines.
- Let $PA = x$, $PB = y$, and $PC = z$. Applying the Law of Cosines on $\triangle PBC$, $\triangle PAB$, and $\triangle PCA$, we get

$$y^2 + 196 - 28y\cos\theta = z^2.$$

$$x^2 + 169 - 26x\cos\theta = y^2,$$

$$z^2 + 225 - 30z\cos\theta = x^2.$$

- Add the equations to obtain

$$(x^2 + y^2 + z^2) + 590 - (26x + 28y + 30z)\cos\theta = (x^2 + y^2 + z^2)$$

- Therefore, $(13x + 14y + 15z)\cos\theta = 295$.

- Note that the area of $\triangle PAB$ is $[PAB] = \frac{1}{2}(13x)\sin\theta$. We get similar relations for the PBC and PCA.
- Adding the areas,

$$[ABC] = [PAB] + [PBC] + [PCA] = \frac{1}{2}(13x + 14y + 15z)\sin\theta = 84.$$

- Thus, $(13x + 14y + 15z)\sin\theta = 168$.
- We can now divide this equation by $(13x + 14y + 15z)\cos\theta = 295$ to determine $\tan\theta = \frac{168}{295}$.
- The answer is $\boxed{463}$.

Triangle Formulas

- There are several formulas that are very useful when computing quantities within a triangle
- Area formulas:

$$[ABC] = \frac{1}{2}ah_a = \frac{1}{2}bc\sin A = \sqrt{s(s-a)(s-b)(s-c)} = rs = \frac{abc}{4R}.$$

- Ratio Lemma: If D is any point on line BC then

$$\frac{BD}{CD} = \frac{AB}{AC} \cdot \frac{\sin BAD}{\sin CAD}.$$

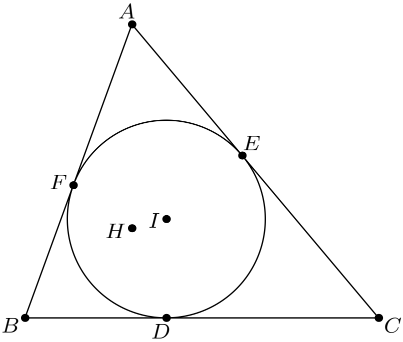
- Stewart's Theorem: Let D be a point on side BC of triangle ABC. If $AD = d,\ BD = m,\ CD = n$ then

$$b^2m + c^2n = a(d^2 + mn).$$

- The following problems, while not falling directly to these formulas, can be computed much more efficiently through use of these and various other relations within a triangle

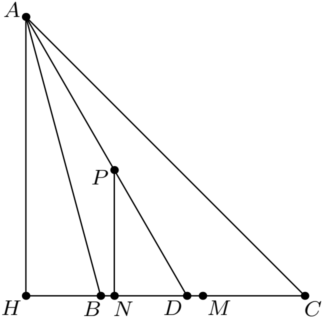
Triangle Formulas

- A couple more formulas, where I is the incenter and H is the orthocenter:
- $AE = AF = s - a$; $BF = BD = s - b$; $CD = CE = s - c$
- $\tan\frac{A}{2} = \frac{r}{s-a}$, $\tan\frac{B}{2} = \frac{r}{s-b}$, $\tan\frac{C}{2} = \frac{r}{s-c}$
- $AH = 2R|\cos A|$, $BH = 2R|\cos B|$, $CH = 2R|\cos C|$



2014 AIME II # 14

In $\triangle ABC$, $AB = 10$, $\angle A = 30^\circ$, and $\angle C = 45^\circ$. Let H, D, and M be points on line $\overline{BC}$ such that $\overline{AH} \perp \overline{BC}$, $\angle BAD = \angle CAD$, and $BM = CM$. Point N is the midpoint of segment $\overline{HM}$, and point P is on ray AD such that $\overline{PN} \perp \overline{BC}$. Then $AP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

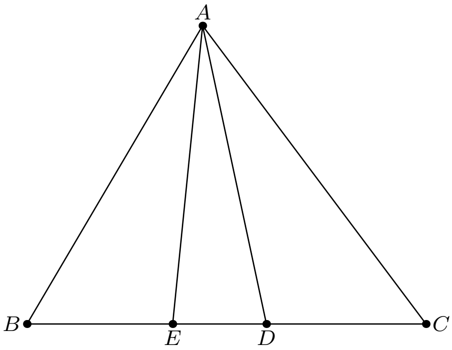


2014 AIME II # 14

- Most angles are nice and easily computable, e.g. $B = 105^\circ$
- How are we going to find AP?
- Note that AH, PN perpendicular to BC
- $\angle ADB = 180^\circ - \angle ABD - \angle BAD = 60^\circ$
- $\triangle ABD, \triangle PND$ are $30^\circ - 60^\circ - 90^\circ$ triangles
- $AP = AD - PD = 2HD - 2ND = 2HN = HM$
- So to find AP we only need to find HM
- Let's split $HM = BH + BM$
- $BH = AB \cos \angle ABH = 10 \cos 75^\circ = 10 \cdot \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{5\sqrt{6}-5\sqrt{2}}{2}$
- Law of Sines: $\frac{BC}{\sin \angle BAC} = \frac{AB}{\sin \angle ACB} \implies BC = 10\frac{\sin 45^\circ}{\sin 30^\circ} = 5\sqrt{2}$
- $BM = \frac{1}{2} BC = \frac{5\sqrt{2}}{2}$
- $HM = \frac{5\sqrt{6}-5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} = \frac{5\sqrt{6}}{2}$, $HM^2 = \frac{75}{2} \implies \boxed{77}$

2005 AIME II # 14

In triangle ABC, $AB = 13$, $BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = \frac{p}{q}$ where p and q are relatively prime positive integers, find q.

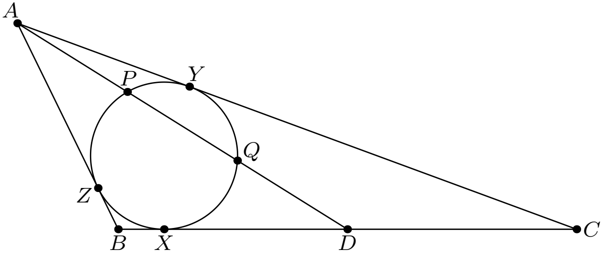


2005 AIME II # 14

- We know lots of lengths and $\angle BAE = \angle CAD$, $\angle BAD = \angle CAE$
- Ratio Lemma!
- We have $\frac{BE}{CE} = \frac{AB}{AC} \cdot \frac{\sin BAE}{\sin CAE}$
- But from angle equalities this is $\frac{AB}{AC} \cdot \frac{\sin CAD}{\sin BAD}$
- Ratio Lemma again to find second factor:
- $\frac{AC}{AB} \cdot \frac{\sin CAD}{\sin BAD} = \frac{CD}{BD} \implies \frac{\sin CAD}{\sin BAD} = \frac{CD}{BD} \cdot \frac{AB}{AC} = \frac{6}{9} \cdot \frac{13}{14} = \frac{13}{21}$
- Plug this in: $\frac{BE}{CE} = \frac{AB}{AC} \cdot \frac{13}{21} = \frac{13}{14} \cdot \frac{13}{21} = \frac{169}{294}$
- Since $BE + CE = BC$ we have $BE + \frac{294}{169} BE = 15$
- Thus $BE = \frac{2535}{463}$
- $\boxed{463}$

2005 AIME I # 15

Triangle ABC has $BC = 20$. The incircle of the triangle evenly trisects the median AD. If the area of the triangle is $m\sqrt{n}$ where m and n are integers and n is not divisible by the square of a prime, find $m + n$.



- Let the median intersect the incircle at P, Q and the incircle touch the sides at X, Y, Z, as shown

2005 AIME I # 15

- Let's let $AP = PQ = QD = x$
- Lines intersecting/tangent to circle $\implies$ power of a point
- $AY^2 = AZ^2 = AP \cdot AQ = x \cdot 2x \implies AY = AZ = x\sqrt{2}$
- Similarly $DX^2 = DP \cdot DQ = x \cdot 2x \implies DX = x\sqrt{2}$
- WLOG $AB \leq AC$ so X is on segment BD
- Clever observation:
  $AZ = DX, BZ = BX \implies AB = AZ + BZ = BX + DX = BD$
- So $AB = \frac{1}{2}BC = 10$
- To find $[ABC]$, it seems promising to find $b = AC$ and use Heron
- This is because we can write several length equations
- $DX = DB - BX = \frac{a}{2} - \frac{a+c-b}{2} = \frac{b-c}{2} = \frac{1}{2}b - 5$
- So $x\sqrt{2} = \frac{1}{2}b - 5, 2x^2 = \left(\frac{1}{2}b - 5\right)^2$

2005 AIME I # 15

- We can use median length formula too:
- $AD^2 = \frac{2b^2+2c^2-a^2}{4} = \frac{b^2-100}{2}$
- $AD = 3x$ so $9x^2 = \frac{b^2-100}{2}$
- Now $x^2 = \frac{1}{2}\left(\frac{1}{2}b-5\right)^2 = \frac{b^2-100}{18}$
- Solve this equation for b to get $b = 10, 26$
- Throw out first solution because $(a, b, c) = (10, 20, 10)$ fails triangle inequality
- So $(a, b, c) = (20, 26, 10)$
- Area $\sqrt{28 \cdot 8 \cdot 2 \cdot 18} = 24\sqrt{14}$
- $\boxed{38}$