Lesson 3: Equations B

Adithya B., Brian L., William W., Daniel X.

April 2020

Homework Review!

Mexico 2011/3

Let n be a positive integer. Find all real solutions $(a_1, a_2, \ldots, a_n)$ to the system:

$$a_1^2 + a_1 - 1 = a_2$$
$$a_2^2 + a_2 - 1 = a_3$$
$$\vdots$$
$$a_n^2 + a_n - 1 = a_1$$

- $a_i(a_i + 1) = a_{i+1} + 1$
- $\prod_{i=1}^{n} a_i(a_i + 1) = \prod_{i=1}^{n}(a_{i+1} + 1) \implies \prod_{i=1}^{n} a_i = 1.$
- Sum the equations: $\sum_{i=1}^{n} a_i^2 = n.$
- $\frac{a_1^2+a_2^2+\cdots+a_n^2}{n} \geq \sqrt[n]{(a_1 a_2 \cdots a_n)^2} = 1.$
- $a_1^2 = a_2^2 = \cdots = a_n^2$. The solutions $(1, 1, 1, \ldots)$ and $(-1, -1, \ldots)$.

Homework Review!

USAMO 2015/1

Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

- $x + y$ is a multiple of 3, so set it to $3a$
- Note that $x^2 + xy + y^2 = (x+y)^2 - (x+y)y + y^2$, so the equation becomes

$$y^2 - 3ay + 9a^2 = (a+1)^3 = a^3 + 3a^2 + 3a + 1$$

- Writing as a quadratic in y, $y^2 - 3ay - (a^3 - 6a^2 + 3a + 1) = 0$

USAMO 2015/1

- The quadratic yields an integer root, so we get that the discriminant has to be a perfect square.
- The discriminant is $\Delta = 9a^2 + 4a^3 - 24a^2 + 12a + 4 = 4a^3 - 15a^2 + 12a + 4$
- This factorizes as $(a-2)^2(4a+1)$, so $4a+1$ has to be a perfect square
- Setting $4a + 1 = 4n^2 + 4n + 1$ gives $a = n^2 + n$
- $\Delta = (n^2 + n - 2)^2(2n+1)^2$, so $y = \frac{3(n^2+n) \pm (n^2+n-2)(2n+1)}{2}$.
- This gives $y = n^3 + 3n^2 - 1, -n^3 + 3n + 1$
- Plugging back in, we have that $x = -n^3 + 3n + 1$ or $n^3 + 3n^2 - 1$
- So, our solution set is $(x, y) = (n^3 + 3n^2 - 1, -n^3 + 3n + 1)$ or its flip.

Radicals

Key ideas

- Try to remove the radicals by raising to the *n*th power
- Try to minimize number of powers you take until all radicals are cleared
- See if any nice substitutions can make the radicals simpler
- Always check back for extraneous solutions

Radicals

2015 MPFG # 17

Let S be the sum of all distinct real solutions of the equation

$$\sqrt{x+2015} = x^2 - 2015.$$

Compute $\lfloor 1/S \rfloor$.

- Notice that the radical is just there to scare you. We can just square the equation
- $x + 2015 = x^4 - 2 \cdot 2015x^2 + 2015^2 \implies$ $x^4 - 2 \cdot 2015x^2 - x + 2015^2 - 2015 = 0$
- Quartics are hard to solve, so let's try to make this a quadratic!
- If 2015 is the variable, we have $2015^2 - 2015(1 + 2x^2) + (x^4 - x) = 0$
- By quadratic formula, $2015 = \frac{1+2x^2 \pm \sqrt{4x^2+4x+1}}{2} = \frac{(1+2x^2) \pm (2x+1)}{2}$

2015 MPFG # 17

- So, either $x^2 + x + 1 = 2015$ or $x^2 - x = 2015$
- (Note that this tells us that the original quartic actually factored as $(x^2 + x - 2014)(x^2 - x - 2015)$, but in this case we got the factorization for free!)
- Using quadratic formula on each, we get $x = \frac{-1 \pm \sqrt{8057}}{2}$ and $x = \frac{1 \pm \sqrt{8061}}{2}$. Do these all work?
- Our only solutions are $\frac{-1-\sqrt{8057}}{2}$ and $\frac{1+\sqrt{8061}}{2}$
- $\lfloor 1/S \rfloor = \left\lfloor \frac{2}{\sqrt{8061}-\sqrt{8057}} \right\rfloor = \left\lfloor \frac{\sqrt{8061}+\sqrt{8057}}{2} \right\rfloor = 89$

Floors

- $\lfloor x \rfloor$ is defined as the greatest integer less than x
- Often times, equations will involve floors
- Floor equations can be solved in many ways, including...
  - Noting that $\lfloor x \rfloor$ is close to x for big x, which gives a bound on what x can be.
  - casework on $\lfloor x \rfloor$
- Substituting $x = \lfloor x \rfloor + \{x\}$ where $\{x\} \in [0, 1)$ is the fractional part of x

Floors

2019 HMMT November Team #5

Compute the sum of all positive real numbers $x \leq 5$ satisfying

$$x = \frac{\lceil x^2 \rceil + \lceil x \rceil \cdot \lfloor x \rfloor}{\lceil x \rceil + \lfloor x \rfloor}.$$

- $0 < x \leq 5$, so there aren't that many pairs of $(\lfloor x \rfloor, \lceil x \rceil)$
- First, $\lfloor x \rfloor = \lceil x \rceil$ when x is an integer. Which integers are solutions?
- $1, 2, 3, 4, 5$ are all solutions
- Now, check $(\lfloor x \rfloor, \lceil x \rceil) = (0, 1); (1, 2); (2, 3); (3, 4); (4, 5)$
- $(0, 1)$: Denominator is 1, so RHS is an integer
- $(1, 2)$: $x = \frac{\lceil x^2 \rceil + 2}{3}$. $1 < x^2 < 4$, so we can have the numerator be $4, 5$. Both $x = \frac{4}{3}, \frac{5}{3}$ work

2019 HMMT November Team #5

- $(2,3): x = \frac{\lceil x^2 \rceil + 6}{5}$, $4 < x^2 < 9$. Note that the numerator needs to be between 11 and 14, so we can have $x = \frac{11}{5}, \frac{12}{5}, \frac{13}{5}, \frac{14}{5}$. In fact, they all work.
- So far, it seems that everything is working. Now, in general for $(n, n+1)$, if we let $x = n + \frac{k}{2n+1}$, we have

$$x^2 = n^2 + \frac{2nk}{2n+1} + \frac{k^2}{(2n+1)^2} = n^2 + k - \left(\frac{k}{2n+1}\right) + \left(\frac{k}{(2n+1)}\right)^2$$

- If we let $y = \frac{k}{2n+1}$, note that $0 < y < 1$, so $-1 < y^2 - y < 0$, and hence $\lceil x^2 \rceil = n^2 + k$
- As $x = \frac{(n^2+k)+(n^2+n)}{2n+1}$, this is exactly what we want, so all k work

2019 HMMT November Team #5

- So, our solutions are the integers, and the fractions between n and $n+1$ with denominator $2n+1$ for $n = 1, 2, 3, 4$
- The sum of the fractions with denominator $2n+1$ is $\frac{2n^2+n+1}{2n+1} + \ldots + \frac{2n^2+3n}{2n+1} = 2n^2 + \frac{1+2+\ldots+2n}{2n+1} = 2n^2 + n$
- Our final answer is $1+2+3+4+5+1+2+3+4+2(1^2+2^2+3^2+4^2) = 85$

Floors

2012 HMMT Algebra #5

Find all ordered triplets (a, b, c) of positive reals that satisfy: $\lfloor a \rfloor bc = 3$, $a \lfloor b \rfloor c = 4$, $ab \lfloor c \rfloor = 5$.

- Multiply everything, giving $60 = a^2b^2c^2\lfloor a \rfloor \lfloor b \rfloor \lfloor c \rfloor \geq (\lfloor a \rfloor \lfloor b \rfloor \lfloor c \rfloor)^3$
- Thus, we find that $\lfloor a \rfloor \lfloor b \rfloor \lfloor c \rfloor \leq 3$
- Now at least two of $\lfloor a \rfloor, \lfloor b \rfloor, \lfloor c \rfloor$ are 1.
- Let's do a first case: suppose all three floors are 1.
- Now $bc = 3, ca = 4, ab = 5$: how do we solve this?
- We multiply and take the square root:
  $abc = \sqrt{(ab)(bc)(ca)} = 2\sqrt{15}$.

2012 HMMT Algebra #5

- Divide abc by bc, ca, ab to get a, b, c : $a = \frac{2\sqrt{15}}{3}, b = \frac{2\sqrt{15}}{4}, c = \frac{2\sqrt{15}}{5}$.
- This doesn't satisfy floor though: $\lfloor a \rfloor = 2$.
- Now we take cases: suppose $\lfloor a \rfloor = \lfloor b \rfloor = 1$.
- We'll only do this one case and leave the others as exercises; in all cases, we consider possible values of the third floor, solve the resulting equations, and verify that all floor hypotheses are satisfied.
- Here $\lfloor c \rfloor = 2$ or $\lfloor c \rfloor = 3$ (we've dealt with $\lfloor c \rfloor = 1$ already).
- If $\lfloor c \rfloor = 2$ then $bc = 3, ca = 4, ab = \frac{5}{2} \implies a = \frac{\sqrt{30}}{3}, b = \frac{\sqrt{30}}{4}, c = \frac{2\sqrt{30}}{5}$ and this satisfies our hypotheses $\lfloor a \rfloor = \lfloor b \rfloor = 1, \lfloor c \rfloor = 2$ so it's a solution.
- Otherwise $\lfloor c \rfloor = 3$ so $bc = 3, ca = 4, ab = \frac{5}{3}$.
- Now $a = \frac{2\sqrt{5}}{3}, b = \frac{\sqrt{5}}{2}, c = \frac{6\sqrt{5}}{5}$ but this fails as $\lfloor c \rfloor = 2$ instead of $\lfloor c \rfloor = 3$.

Floors

2012 AIME II #10

Find the number of positive integers n less than 1000 for which there exists a positive real number x such that $n = x\lfloor x \rfloor$.

- We can consider the equation as the form $x = \frac{n}{\lfloor x \rfloor}$
- Suppose we let $k = \lfloor x \rfloor$
- Then, we see that $k \leq x < k + 1$ by properties of floors, so we need $k \leq \frac{n}{k} < k + 1$.
- We have $k^2 \leq n < k(k + 1)$.
- Note that all n of this form work, so we just count how many of these there are.
- For each k, there are k values, so we have our answer is $1 + 2 + \cdots + 31 = 496$.

2012 AIME II #10

-

Floors

Adithya Balachandran

Let a be a real number such that

$$\lfloor a \rfloor^4 - 4a^2 + \lfloor a \rfloor - 1 = 0.$$

Then, the sum of the squares of all such real numbers a can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the value of $m + n$.

- $\lfloor a \rfloor$ is an integer, so we will try to isolate terms with $\lfloor a \rfloor$ on one side.
- Equation rearranges to $4a^2 = \lfloor a \rfloor^4 + \lfloor a \rfloor - 1$.
- $a^2 = \frac{\lfloor a \rfloor^4 + \lfloor a \rfloor - 1}{4} \implies a = \pm \frac{\sqrt{\lfloor a \rfloor^4 + \lfloor a \rfloor - 1}}{2}$.
- The reason that this relation is useful is that when a is large, the right side is approximately $\lfloor a \rfloor^2$, which is much larger than the left side.
- Let's first look at what happens when we take the positive root.

Adithya Balachandran

- Since $0 \leq a - \lfloor a \rfloor < 1$,

$$0 \leq \frac{\sqrt{\lfloor a \rfloor^4 + \lfloor a \rfloor - 1}}{2} - \lfloor a \rfloor < 1.$$

- The only value of $\lfloor a \rfloor$ that can work in this case is $\lfloor a \rfloor = 2$.
- This yields the solution $a = \frac{\sqrt{17}}{2}$.
- Now, we will do the case for the negative root $(a < 0)$. Since $0 \leq a - \lfloor a \rfloor < 1$,

$$0 \leq -\frac{\sqrt{\lfloor a \rfloor^4 + \lfloor a \rfloor - 1}}{2} - \lfloor a \rfloor < 1.$$

- The only solution in this case is $\lfloor a \rfloor = -2$, or $a = -\frac{\sqrt{13}}{2}$.
- The sum of the squares of the solutions is $\frac{17}{4} + \frac{13}{4} = \frac{15}{2}$, so the answer is 17.

Exponents and Logarithms

Familiar Properties:

1. Exponent addition when multiplying two numbers with the same base: $a^x \cdot a^y = a^{x+y}$
2. Exponent subtraction when dividing two numbers with the same base: $\frac{a^x}{a^y} = a^{x-y}$.
3. Multiplication with the same exponent: $a^x \cdot b^x = (ab)^x$.
4. Logarithm addition: $\log_b x + \log_b y = \log_b(xy)$.
5. Logarithm subtraction: $\log_b x - \log_b y = \log_b\left(\frac{x}{y}\right)$.
6. Change of base: $\frac{\log_b c}{\log_a c} = \log_b a$.
7. Logarithm power rules: $\log_b x^a = a \log_b x$ and $\log_{b^a} x = \frac{1}{a} \log_b x$.
8. Reciprocal of a logarithm: $\frac{1}{\log_b a} = \log_a b$.

Exponents and Logarithms

- Logarithm questions are often just normal polynomial system of equations disguised with the notation. The key is to make substitutions to transform them

Logarithms

2002 AIME I #6

The solutions to the system of equations

$$\log_{225} x + \log_{64} y = 4$$
$$\log_x 225 - \log_y 64 = 1$$

are (x_1, y_1) and (x_2, y_2). Find $\log_{30}(x_1 y_1 x_2 y_2)$.

- We see 225 and 64 are both perfect powers, so we can use logarithm properties to rewrite the equations:

$$\frac{1}{2}\log_{15} x + \frac{1}{6}\log_2 y = 4, \qquad 2\log_x 15 - 6\log_y 2 = 1$$

- Let $a = \log_{15} x$ and $b = \log_2 y$, so the equations are $\frac{1}{2}a + \frac{1}{6}b = 4$ and $\frac{2}{a} - \frac{6}{b} = 1$. We can now solve for a and b.

2002 AIME I #6

- We can rewrite the equation $\frac{1}{2}a + \frac{1}{6}b = 4$ as $b = 24 - 3a$. We can also multiply the equation $\frac{2}{a} - \frac{6}{b} = 1$ by ab to obtain $2b - 6a = ab$.
- Substituting for b, $2(24 - 3a) - 6a = a(24 - 3a)$. We can rewrite this equation as $3a^2 - 36a + 48 = 0$ or $a^2 - 12a + 16 = 0$.
- Remember that the solutions to this equation are $\log_{15} x_1$ and $\log_{15} x_2$.
- Therefore, $\log_{15}(x_1 x_2) = \log_{15} x_1 + \log_{15} x_2 = 12$ by Vieta's, so $x_1 x_2 = 15^{12}$.
- As we had $b = 24 - 3a$, $\log_2 y_1 = 24 - 3\log_{15} x_1$ and $\log_2 y_2 = 24 - 3\log_{15} x_2$.
- Therefore, $\log_2(y_1 y_2) = \log_2 y_1 + \log_2 y_2 = 24 - 3\log_{15} x_1 + 24 - 3\log_{15} x_2 = 48 - 3(\log_{15} x_1 + \log_{15} x_2) = 12$.
- This means $y_1 y_2 = 2^{12}$, so $\log_{30}(x_1 x_2 y_1 y_2) = \log_{30}(2^{12} 15^{12}) = \log_{30}(30^{12}) = 12$.

Logarithms

2016 PUMaC Algebra #3

For positive real numbers x and y, let $f(x,y) = x^{\log_2 y}$. Compute the sum of the solutions to the equation

$$4096 f(f(x,x),x) = x^{13}$$

- Set $x = 2^k$.
- $f(x,x) = (2^k)^k = 2^{k^2}$.
- $f(f(x,x),x) = (2^{k^2})^k = 2^{k^3}$.
- Now need $2^{k^3+12} = 2^{13k}$.
- Thus $k^3 + 12 = 13k$.
- Note that $k = 1$ is a solution and factor $k^3 - 13k + 12 = (k-1)(k^2+k-12) = (k-1)(k-3)(k+4)$.
- Thus the sum of all x is $2^1 + 2^3 + 2^{-4} = \frac{161}{16}$

2016 PUMaC Algebra #3

-

Logarithms

2017 AIME I #14

Let $a > 1$ and $x > 1$ satisfy $\log_a(\log_a(\log_a 2) + \log_a 24 - 128) = 128$ and $\log_a(\log_a x) = 256$. Find the remainder when x is divided by 1000.

- Undo one layer of logs in the first equation.
- $\log_a(\log_a 2) + \log_a(24) - 128 = a^{128}$.
- Let $a = 2^b$.
- $\log_{2^b}(\log_{2^b} 2) + \log_{2^b}(24) - 128$.
- $\log_{2^b}\left(\frac{1}{b}\right) + \log_{2^b}(24) - 128$
- $\log_{2^b}\left(\frac{24}{b2^{128b}}\right)$
- This equals $a^{128} = 2^{128b}$.
- So $2^{128b} = \log_{2^b}\left(\frac{24}{b2^{128b}}\right)$

2017 AIME I #14

- Undo another layer of logs:
- $(2^b)^{2^{128b}} = 2^{b2^{128b}} = \frac{24}{b2^{128b}}$.
- Let $c = b \cdot 2^{128b}$.
- $2^c = \frac{24}{c}$.
- $c = 3$ is a solution: LHS increasing, RHS decreasing $\implies$ 3 only solution.
- $b \cdot 2^{128b} = 3$.
- Need b to be three times a power of 2: $b = 3 \cdot 2^k$.
- $3 \cdot 2^k \cdot 2^{3 \cdot 2^k \cdot 128} = 3 \implies 2^{k+384 \cdot 2^k} = 1 \implies -k = 384 \cdot 2^k$.
- $b = 3 \cdot 2^{-6} = \frac{3}{64}$.
- $a = 2^{\frac{3}{64}}$.

2017 AIME I #14

- Now look at second equation:
- $\log_a(\log_a x) = 256 \implies x = a^{a^{256}}$
- $x = a^{2^{256 \cdot \frac{3}{64}}} = a^{2^{12}} = 2^{2^{12} \cdot \frac{3}{64}} = 2^{192}$.
- Answer extraction: we know that $x \equiv 0 \pmod 8$ so just need $x \pmod{125}$.
- By Euler's Theorem $2^{100} \equiv 1 \pmod{125}$.
- $\phi(125) = 100$.
- $2^{192} \equiv 2^{-8} \equiv \frac{1}{256} \equiv \frac{1}{6} \equiv 21 \pmod{125}$
- Now $x \equiv 896 \pmod{1000}$ using Chinese Remainder Theorem.