Lesson 2: Equations

Adithya B., Brian L., William W., Daniel X.

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Homework Review!

Find the remainder when $x^{28}+1$ is divided by $x^4+x^3+x^2+x+1$.

- $x^{28}+1$ divided by $\frac{x^5-1}{x-1}$
- $x^{28}+1=(x^5-1)Q(x)+R(x)$
- $x^{28}+1=(x^{28}-x^{23})+x^{23}+1$
- $x^{23}-x^{18}+x^{18}+1$
- $x^{18}+1$
- x^8+1
- x^3+1
- $x^{28}+1=(x^{28}-x^{23})+(x^{23}-x^{18})+\ldots+(x^3+1)$

Homework Review!

ISL 2005/N3

Let a, b, c, d, e, f be positive integers and let $S = a + b + c + d + e + f$. Suppose that the number S divides $abc + def$ and $ab + bc + ca - de - ef - df$. Prove that S is composite.

- $P(x) = (x + a)(x + b)(x + c)$, $Q(x) = (x - d)(x - e)(x - f)$
- $P(x) - Q(x) =$ $(a+b+c+d+e+f)x^2 + (ab+bc+ca-de-ef-df)x + (abc+def)$
- $P(f) - Q(f) = (a + f)(b + f)(c + f) - 0$
- $S|(a + f)(b + f)(c + f)$
- $S >$ all the factors, so it can't divide any of them
- S can't be prime or 1 so S must be composite

What are Equations?

- This is probably going to be our broadest topic (let us know after class if we should have a second class on it.)
- Equations are sets of equalities from which we can get tuples of variables which satisfy all the equalities. Examples include systems of linear equations, or the polynomials we covered last class.
- Since equations are so broad, it's hard to give any real specific definitions or methodologies. We instead will give a broad range of problems involving equations and demonstrate motivation.

Polynomials

Brian Liu

Solve $64x^6 + 64x^5 - 320x^4 - 448x^3 - 80x^2 + 4x + 1 = 0$.

- We want to decrease the coefficients somehow to make them more manageable, and we see that $y = 2x$ will simplify things
- $y^6 + 2y^5 - 20y^4 - 56y^3 - 20y^2 + 2y + 1 = 0$
- $\left(y^3 + \frac{1}{y^3}\right) + 2\left(y^2 + \frac{1}{y^2}\right) - 20\left(y + \frac{1}{y}\right) - 56 = 0$
- Each of these terms is a function of $y + \frac{1}{y}$
- $z = y + \frac{1}{y}$
- $(z^3 - 3z) + 2(z^2 - 2) - 20z - 56 = 0$
- $z^3 + 2z^2 - 23z - 60 = 0$
- $(z + 3)(z + 4)(z - 5) = 0$
- $y = \frac{-3\pm\sqrt{5}}{2}, -2 \pm \sqrt{3}, \frac{5\pm\sqrt{21}}{2}$
- $x = \frac{-3\pm\sqrt{5}}{4}, \frac{-2\pm\sqrt{3}}{2}, \frac{5\pm\sqrt{21}}{4}$.

2000 AIME II # 13

The equation $2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$ has exactly two real roots, one of which is $\frac{m+\sqrt{n}}{r}$, where m, n and r are integers, m and r are relatively prime, and $r > 0$. Find $m + n + r$.

- Note that the middle 3 terms form a geometric series, and so it'd be nice if we could try to take advantage of this structure.
- $2000x^6 - 2 + x\frac{1000x^6-1}{10x^2-1} = 0$
- $2(1000x^6 - 1) + x\frac{1000x^6-1}{10x^2-1} = 0$
- Now, we see that we can factor out $\frac{1000x^6-1}{10x^2-1} = 100x^4 + 10x^2 + 1$
- $(100x^4 + 10x^2 + 1)(20x^2 + x - 2) = 0$
- Now, see that $100x^4 + 10x^2 + 1$ has no real roots, and $20x^2 + x - 2$ has real roots at $x = \frac{-1\pm\sqrt{161}}{40}$
- Answer $= -1 + 161 + 40 = 200$

Combining Equations

1990 AIME #15

If a, b, x and y are real numbers such that $ax + by = 3$, $ax^2 + by^2 = 7$, $ax^3 + by^3 = 16$, and $ax^4 + by^4 = 42$, find $ax^5 + by^5$.

- How do we get $ax^{n+1} + by^{n+1}$ from $ax^n + by^n$?
- Let's try multiplying $ax^n + by^n$ by $x + y$.
- $(ax^n + by^n)(x + y) = ax^{n+1} + by^{n+1} + xy(ax^{n-1} + by^{n-1})$.
- Rearrange: $ax^{n+1} + bx^{n+1} = (x + y)(ax^n + by^n) - xy(ax^{n-1} + bx^{n-1})$
- Let $s = x + y$ and $p = xy$.
- For $n = 2$, we have $16 = 7s - 3p$. For $n = 3$, we have $42 = 16s - 7p$.
- Solve to get $s = -14$ and $p = -38$.
- $ax^5 + by^5 = -14(ax^4 + by^4) + 38(ax^3 + by^3) = -14 \cdot 42 + 38 \cdot 16 = 20$.

Combining Equations

Mildorf Mock AIME

a, b,and c are complex numbers such that

$$\begin{aligned} a+b+c &= 1 \\ a^2+b^2+c^2 &= 3 \\ a^3+b^3+c^3 &= 7 \end{aligned}$$

Find $a^7+b^7+c^7$.

- Let $S_n = a^n + b^n + c^n$. We are given $S_1 = 1, S_2 = 3$, and $S_3 = 7$.
- Can we use a factorization to obtain S_7?
- Multiply S_n by $a+b+c$:

$$(a+b+c)S_n = S_{n+1} + (a^n b + a^n c + b^n a + b^n c + c^n a + c^n b).$$

- The last few terms look like they can be formed by multiplying S_{n-1} by $ab+bc+ca$.

- $(a^{n-1}+b^{n-1}+c^{n-1})(ab+bc+ca) =$
  $(a^nb+a^nc+b^na+b^nc+c^na+c^nb)+(a^{n-1}bc+ab^{n-1}c+abc^{n-1})$
- The left side is $(ab+bc+ca)S_{n-1}$. The first group on the right side is $(a+b+c)S_n - S_{n+1}$. The second group on the right is $abcS_{n-2}$.

$$(ab+bc+ca)S_{n-1} = (a+b+c)S_n - S_{n+1} + abcS_{n-2},$$

$$S_{n+1} = (a+b+c)S_n - (ab+bc+ca)S_{n-1} + abcS_{n-2}.$$

- $ab+bc+ca = \frac{(a+b+c)^2-(a^2+b^2+c^2)}{2} = \frac{1-3}{2} = -1.$
- Use the factorization
  $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$
- We have $7-3abc = 1(3+1) = 4$, so $abc = 1$.
- The recursion is $S_{n+1} = S_n + S_{n-1} + S_{n-2}$.
- We can calculate $S_4 = 11$, $S_5 = 21$, $S_6 = 39$, and $S_7 = 71$.

Combining Equations

2014 HMMT Algebra #9

Given that a, b, c are complex numbers satisfying

$$a^2 + ab + b^2 = 1 + i$$
$$b^2 + bc + c^2 = -2$$
$$c^2 + ca + a^2 = 1$$

compute $(ab + ac + bc)^2$.

- We want to try to combine the three equations to get $(ab + ac + bc)^2$. Let's first expand it.
- $(ab + ac + bc)^2 = a^2b^2 + a^2c^2 + b^2c^2 + 2abc(a + b + c)$
- Let's try to get the a^2bc terms first. To do, this we can just multiply two equations together
- $(b^2 + bc + c^2)(c^2 + ca + a^2) =$ $a^2b^2 + a^2c^2 + b^2c^2 + c^4 + bc^3 + ac^3 + (a^2bc + ab^2c + abc^2)$

HMMT Algebra #9

- In order to preserve symmetry, we compute the other two sums and add them to this one to get: $(a^4 + b^4 + c^4) + 3(a^2b^2 + a^2c^2 + b^2c^2) + (a^3b + b^3a + a^3c + c^3a + b^3c + c^3b) + 3(a^2bc + ab^2c + abc^2)$
- Now, we get a few terms we don't want though. For example, c^4 and bc^3 both don't appear in our final expression. However, note that these both appear in $(a^2 + ab + b^2)^2$. So, let's compute that
- $(a^2 + ab + b^2)^2 = a^4 + b^4 + 2a^3b + 2ab^3 + 3a^2b^2$
- Similarly, we add the other two symmetric expansions to get $(a^2 + ab + b^2)^2 + (a^2 + ac + c^2)^2 + (b^2 + bc + c^2)^2 = 2(a^4 + b^4 + c^4) + 2(a^3b + b^3a + a^3c + c^3a + b^3c + c^3b) + 3(a^2b^2 + a^2c^2 + b^2c^2)$
- If we multiply the first expression by 2 and subtract the second expression, all the terms we don't want disappear!
- In fact, we get $3(a^2b^2 + a^2c^2 + b^2c^2) + 6(a^2bc + ab^2c + abc^2) = 3(ab + ac + bc)^2$

HMMT Algebra #9

- So, our answer is $\frac{2((1+i)(-2)+(1+i)(1)+(-2)(1))-((1+i)^2+(-2)^2+(1)^2)}{3} = -\frac{11}{3} - \frac{4}{3}i$

More Systems

2005 Iberoamerican/1

Determine all triples of real numbers (x, y, z) such that

$$\begin{aligned} xyz &= 8 \\ x^2y + y^2z + z^2x &= 73 \\ x(y-z)^2 + y(z-x)^2 + z(x-y)^2 &= 98. \end{aligned}$$

- The last expression looks ugly, so let's expand it: $x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 - 6xyz = 98$
- Substituting in what we already know, this gives $xy^2 + yz^2 + zx^2 = 73$, so $xy^2 + yz^2 + zx^2 = x^2y + y^2z + z^2x$
- Move everything to one side: does this remind you of anything? $x^2y + y^2z + z^2x - xy^2 - yz^2 - zx^2 = 0$

2005 Iberoamerican/1

- The left hand side is actually factorizable! It is $(x-y)(x-z)(y-z)=0$, so two of the variables are equal
- WLOG $x=y$. Then, our equations become $x^2z=8$, $x^3+x^2z+z^2x=x^3+8+z^2x=73$
- Now, solve normally. The second expression gives $z^2=\frac{65}{x}-x^2$, so substitute it back in the first equation to get $65x^3-x^6=64$
- This is a quadratic in x^3. We get $x^3=1,64 \implies x=1,4$
- Our final solutions are $(1,1,8)$, $\left(4,4,\frac{1}{2}\right)$, and cyclic permutations

Algebraic Manipulation

2014 AIME II #14

Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4$$

There are positive integers a, b, and c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a + b + c$.

- Each term on the left-hand side is $\frac{n}{x-n}$: add 1 to get $\frac{x}{x-n}$
- $\frac{x}{x-3} + \frac{x}{x-5} + \frac{x}{x-17} + \frac{x}{x-19} = x^2 - 11x$
- $x = 0$ is not largest solution $\implies \frac{1}{x-3} + \frac{1}{x-5} + \frac{1}{x-17} + \frac{1}{x-19} = x - 11.$
- $3, 5, 17, 19$ are symmetric around 11 $\implies$ let $y = x - 11.$
- $\frac{1}{y-8} + \frac{1}{y-6} + \frac{1}{y+6} + \frac{1}{y+8} = y$

2014 AIME II #14

- $\frac{2y}{y^2-64} + \frac{2y}{y^2-36} = y$
- $\frac{2}{y^2-64} + \frac{2}{y^2-36} = 1$
- let $z = y^2 \implies \frac{2}{z-64} + \frac{2}{z-36} = 1$
- $2(z-36) + 2(z-64) = (z-64)(z-36)$
- $4z - 200 = z^2 - 100z + 2304$
- $z^2 - 104z + 2504 = 0 \implies z = 52 + \sqrt{200}$
- $y = \sqrt{z} = \sqrt{52 + \sqrt{200}}$
- $x = 11 + y = 11 + \sqrt{52 + \sqrt{200}}$
- $11 + 52 + 200 = 263$