Lesson 17: Counting B

Adithya B., Brian L., William W., Daniel X.

November 2020

Principle of Inclusion and Exclusion

- Often times, we can't break a question into nice, disjoint cases
- PIE gives a way of dealing with cases which overlap

PIE

Given sets $A_1, A_2, \ldots, A_k$, we have

$$|A_1 \cup A_2 \cup \ldots \cup A_k| = \sum_{S \subseteq [k]} (-1)^{1+|S|} \left| \bigcap_{s \in S} A_s \right|$$

- This notation may look complicated, so let's break it down for the 2 and 3 variable cases:
  - $k = 2$: $|A \cup B| = |A| + |B| - |A \cap B|$
  - $k = 3$: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
  - You can visualize as a Venn Diagram

Principle of Inclusion and Exclusion

- Let's prove PIE!
- We do so with induction on k. The base case of $k = 2$ is clear, so let's move to the inductive step
- Suppose PIE is true for $k - 1$. To get to k, note that

$$|A_1 \cup A_2 \cup \ldots \cup A_k| = |(A_1 \cup A_2 \cup \ldots \cup A_{k-1}) \cup A_k|$$

- Using the result of PIE at $k = 2$, we get this is

$$|A_1 \cup \ldots \cup A_{k-1}| + |A_k| - |A_1 \cup \ldots \cup A_{k-1} \cap A_k|$$

- We already know the first term by induction. How can we get the last term?
- To get the last term, take the representation for $k - 1$, and tack on an A_k in every term

Principle of Inclusion and Exclusion

- In particular,

$$|A_1 \cup \ldots \cup A_{k-1} \cap A_k| = \sum_{S \subseteq [k-1]} (-1)^{1+|S|} \left| A_k \cap \bigcap_{s \in S} A_s \right|$$

This sums over all subsets of $[k]$ which contain k but are not $\{k\}$

- So, the sum of the last 2 terms sums over

$$\sum_{S \subseteq [k], k \in S} (-1)^{k-|S|} \left| \bigcap_{s \in S} A_s \right|$$

- Adding this to the first term, which sums over all subsets of $[k]$ not including k, we get the entire sum

Principle of Inclusion and Exclusion

- So,

$$|A_1 \cup \ldots \cup A_k| = \sum_{S \subseteq [k], k \notin S} (-1)^{1+|S|} \left| \bigcap_{s \in S} A_s \right| + \sum_{S \subseteq [k], k \in S} (-1)^{1+|S|} \left| \bigcap_{s \in S} A_s \right|$$

$$= \sum_{S \subseteq [k]} (-1)^{1+|S|} \left| \bigcap_{s \in S} A_s \right|$$

  as desired

- In practice, we don't always have to write our counting in terms of A_is. Oftentimes, we can construct our PIE using a very intuitive sense of over and undercounting. We'll see this in the next example

Principle of Inclusion and Exclusion

Folklore

For integers $n > m$, compute the sum

$$\sum_{i=1}^{n} i^m \binom{n}{i} (-1)^{n-i}$$

- Let's expand the sum and try to get a sense of what it means
- $\binom{n}{n} n^m - \binom{n}{n-1}(n-1)^m + \binom{n}{n-2}(n-2)^m - \cdots$
- Each term involves raising something to the power of m, suggesting that we want to look at coloring $\{1, 2, \ldots, m\}$ with n colors
- For instance, the first term is exactly the number of ways to color $\{1, 2, \ldots, m\}$ with n colors without restrictions
- How to interpret the second term?
- Choose $n-1$ colors from the n colors and use them to color $\{1, 2, \ldots, m\}$

Principle of Inclusion and Exclusion

- With the addition, subtraction, and combinatorial interpretation, let's try to view this from the angle of PIE
- We claim that this counts the number of colorings of m objects with n colors such that each color is used at least once
- The first term represents the total number of colorings
- In the second term, we subtract all colorings which don't use a color. However, we oversubtract the cases which miss 2 colors
- So, in the third term, we add back the cases which miss 2 colors
- In this way, we construct an alternating sum which eventually evaluates to the total number of ways to perform this coloring. Note that this idea is analogous to PIE
- As $n > m$, it is impossible to color the blocks with all n colors. So, the sum is $\boxed{0}$

Principle of Inclusion and Exclusion

Mandelbrot, modified

Suppose N is the number of ways to partition the counting numbers from 1 to 15 (inclusive) into five sets with three numbers in each set so that the product of the numbers in each set is divisible by 6. What is the number formed by the first three digits of N?

- What is one thing we can immediately say about the five sets?
- Each will have a multiple of 3.
- The sets are $\{3, \_, \_\}, \{6, \_, \_\}, \{9, \_, \_\}, \{12, \_, \_\}, \{15, \_, \_\}$.
- Note that the second and fourth sets are already divisible by 6.
- There are 5 even numbers and 5 odd numbers remaining.
- Let's try complementary counting now.
- First, let's find how many ways are there such that the product of the elements in $\{3, \_, \_\}$ is not divisible by 6.
- How can we distribute the even numbers among the other 4 sets?

Mandelbrot, modified

- Either a permutation of $(2, 2, 1, 0)$ or $(2, 1, 1, 1)$.
- Consider $(2, 2, 1, 0)$.
- The number of ways to choose the even numbers is $\binom{4}{2}\binom{2}{1}\binom{5}{2}\binom{3}{2} = 360$.
- The number of ways to choose the odd numbers is $\binom{5}{2}\binom{3}{2} = 30$. The total number of ways in this case is $360 \cdot 30 = 10800$.
- Consider $(2, 1, 1, 1)$.
- The number of ways here is $\left(\binom{4}{1}\binom{5}{2} \cdot 3!\right)\left(\binom{5}{2} \cdot 3!\right) = 14400$.
- The number of ways $\{3, \_, \_\}$ is not divisible by 6 is 25200. There are the same number of ways when $\{9, \_, \_\}$ and $\{15, \_, \_\}$ are not divisible by 6. This yields $3 \cdot 25200$ in total.
- We have to subtract the overcount.
- Consider when $\{3, \_, \_\}$ and $\{9, \_, \_\}$ are not divisible by 6. How can we distribute the 5 even numbers among the 3 remaining sets?

Mandelbrot, modified

- Has to be $(2, 2, 1)$.
- The number of ways to choose the even numbers is $3 \cdot \binom{5}{2}\binom{3}{2} = 90$.
- The number of ways to choose the odd numbers is $\binom{5}{2}\binom{3}{2} = 30$.
- 2700 ways in this case.
- We have to subtract by $3 \cdot 2700$ as there are 3 such intersections.
- Do we need to add anything back now? Is it possible for all of the sets $\{3, \_, \_\}$,$\{9, \_, \_\}$, and $\{15, \_, \_\}$ to not be divisible by 6?
- No, because then there would be only 4 spots for 5 even numbers.
- The number of ways that don't work are $3 \cdot 25200 - 3 \cdot 2700 = 67500$.
- The total number of ways is $\binom{10}{2,2,2,2,2} = 113400$.
- $N = 113400 - 67500 = \boxed{459}00$.

Ad Hoc

- Most of the time, combinatorics is hard, and there are no premade strategies which can help simplify it
- In these cases, we want to play around with the question to understand what's going on
  - Try small cases
  - Try special cases which are easy to compute
  - Try to find some patterns

Ad Hoc

2018 MMATHS #9

Diane has a collection of weighted coins with different probabilities of landing on heads, and she flips nine coins sequentially according to a particular set of rules. She uses a coin that always lands on heads for her first and second flips, and she uses a coin that always lands on tails for her third flip. For each subsequent flip, she chooses a coin to flip as follows: if she has so far flipped a heads out of b total flips, then she uses a coin with an $\frac{a}{b}$ probability of landing on heads. What is the probability that after all nine flips, she has gotten six heads and three tails?

- At first glance, this process seems very random and unmotivated. What should we try?
- Let's do a few small cases!

2018 MMATHS #9

- What's the probability that Diane gets $HHTHTHTHH$ (a random string with 6 Hs and 3 Ts)
- $$\frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{3}{5} \cdot \frac{2}{6} \cdot \frac{4}{7} \cdot \frac{5}{8}$$
- Let's do this again with $HHTTTHHHH$
- $$\frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{2}{5} \cdot \frac{3}{6} \cdot \frac{4}{7} \cdot \frac{5}{8}$$
- Notice anything interesting?
- These two values are the same! The denominator is always $\frac{8!}{2}$ and the numerator is $2! \cdot 5!$. Can we prove this in general?
- What is the denominator of the nth flip for $n \geq 4$
- It is always $n - 1$

2018 MMATHS #9

- Consider the flip which produces the ith head. What is the numerator of the corresponding probability?
- It is $i-1$, since $i-1$ heads occurred before it
- So, if we only look at the numerators corresponding to heads, they're some permutation of $1, 1, 2, 3, 4, 5$
- Similarly, the numerators corresponding to tails are some permutation of $1, 1, 2$
- So, no matter the ordering of Hs and Ts, our answer is always $\frac{2!\cdot 5!}{8!/2}$
- So, the probability is

$$\binom{6}{2} \cdot \frac{2! \cdot 5!}{\frac{8!}{2}} = 15 \cdot \frac{1}{4 \cdot \binom{7}{2}} = \frac{15}{84} = \frac{5}{28}$$

Ad Hoc

2017 HMMT C7

There are 2017 frogs and 2017 toads in a room. Each frog is friends with exactly 2 toads. Let N be the number of ways to pair each frog with a toad who is its friend, such that no toad is paired with more than one frog. Compute the number of possible distinct values of N.

- Obviously, 0 is possible, so we assume that N is nonzero
- What if a toad only has one friend?
- Every toad must be paired, so we have a forced pair. Then, we can just remove that pair as it doesn't matter for computing N
- Hence, we arrive at the more general question:
  Given k frogs and k toads such that each frog is friends with exactly 2 toads and each toad is friends with > 1 frogs, compute the number of ways to pair.

2017 HMMT C7

- How many friends does each toad have?
- Actually, each toad has exactly 2 friends since there are exactly 4034 friendships
- Now, if we draw a graph between the frogs and the toads, how does it look?
- Every vertex has 2 neighbors, so we get a disjoint union of multiple cycles
- Consider a specific cycle $T_1 - F_1 - T_2 - \ldots - T_i - F_i$. What happens if we force T_1 to pair with F_1?
- The entire cycle becomes forced. So, each cycle has exactly 2 assignments, either T_1, F_1 or T_1, F_i
- As cycles are disjoint, this means that $N = 2^c$, where c is the number of cycles. What can c be?

2017 HMMT C7

- Each cycle has at least 4 elements, so $c \leq \frac{4034}{4} \implies c \leq 1008$
- So, $N = 2^i$ for any $1 \leq i \leq 1008$, or $N = 0$, and our answer is $\boxed{1009}$