Lesson 16: Counting A

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Casework: General Ideas

- Casework is unappealing, but sometimes it's the only way to solve a question, or at least the easiest way without being extremely clever
- Always try to look for ways to simplify casework. Small observations go a long way
- Have a rough estimate for how long casework will take you before you begin to make sure it's feasible
- Organize your casework. Label every case on your scratchwork to make sure that it's exhaustive and you're not overcounting
- Outline all major cases before beginning to compute any particular case

Casework

2019 HMMT C9

How many ways can one fill a 3×3 square grid with nonnegative integers such that no *nonzero* integer appears more than once in the same row or column and the sum of the numbers in every row and column equals 7?

- How do we begin?
- List out all ways the fill a single row/column
  - $(0,0,7),(0,1,6),(0,2,5),(0,3,4),(1,2,4)$
- What happens if there's a 7 in the grid?
- There must be a $(0,0,7)$ column and row, so we end up wanting to fill up a 2×2
- We can have either 3 7s or 1 7 in the grid
- $6+9 \cdot 6=60$

2019 HMMT C9

- What happens if we have a 6 in the grid?
- $\begin{bmatrix} 0 & 1 & 6 \\ ? & ? & 1 \\ ? & ? & 0 \end{bmatrix}$
- Similarly, we can either have 3 6s, or 1 6. How many in each case?
- $12 + 9 \cdot 2^2 \cdot 2 = 84$
- What happens if we have a 5 in the grid?
- $\begin{bmatrix} 0 & 2 & 5 \\ ? & ? & 2 \\ ? & ? & 0 \end{bmatrix}$
- Same deal - we can either have 3 5s or 1 5. This time the cases are slightly different
- $12 + 9 \cdot 2^2 = 48$
- Proceeding similarly, if there is a 3 there are 12 ways, and if there are only $1, 2, 4$ there are 12 ways
- $60 + 84 + 48 + 12 + 12 = \boxed{216}$

2019 HMMT C9: An Easier Approach

- Remember that our tuples are $(0,0,7),(0,1,6),(0,2,5),(0,3,4),(1,2,4)$. Try writing them in binary
- Key idea: In each tuple, there's a 1 in the 1s place, a 1 in the 2s place, a 1 in the 4s place
- So, we can construct all possible squares by just adding 1 to 3 squares, then 2 to 3 squares, then 4 to 3 squares
- The number of ways to choose 3 squares to add 1 to is $3! = 6$
- Same for the others, so our answer is $6^3 = \boxed{216}$

Casework

2018 AIME II #15

Find the number of functions f from $\{0, 1, 2, 3, 4, 5, 6\}$ to the integers such that $f(0) = 0, f(6) = 12$, and

$$|x - y| \leq |f(x) - f(y)| \leq 3|x - y|$$

for all x and y in $\{0, 1, 2, 3, 4, 5, 6\}$.

- Let $y = x + 1$. We obtain $1 \leq |f(x + 1) - f(x)| \leq 3$. Let $f(x + 1) - f(x) = g(x)$.
- Note that $g(0) + g(1) + g(2) + g(3) + g(4) + g(5) = f(6) - f(0) = 12$
- Given $f(0) = 0$ and $f(6) = 12$, how many times can $g(x)$ be negative (where $0 \leq x \leq 5$)?
- At most once. If it decreases more than that, then $f(6) \leq 3 \cdot 4 - 2 = 10$, which is not true.

2018 AIME II #15

- Casework on the number of times $g(x)$ is negative
- Case 1: $g(x)$ is always positive
- How many ways are there to choose values for $g(x)$ given $1 \leq g(x) \leq 3$?
- For $0 \leq x \leq 5$, let $g(x) = 1$ a times, $g(x) = 2$ b times, and $g(x) = 3$ c times
- $a + 2b + 3c = 12$ and $a + b + c = 6$
- $b + 2c = 6$
- $(0, 6, 0), (1, 4, 1), (2, 2, 2), (3, 0, 3)$
- Number of possible ways to order is $\binom{6}{0,6,0} + \binom{6}{1,4,1} + \binom{6}{2,2,2} + \binom{6}{3,0,3} = 1 + 30 + 90 + 20 = 141$
- Case 2: $g(x)$ is negative once
- Note $2 \leq |f(x+2) - f(x)| \leq 6$ or $2 \leq |g(x) + g(x+1)| \leq 6$
- If $g(x)$ or $g(x+1)$ is negative (only one can be), then it must be equal to -1, and the other has to be equal to 3.

2018 AIME II #15

- If $g(0) = -1$, then $g(1) = 3$. For $g(x)$ where $2 \leq x \leq 5$, we can find the possible values with a similar procedure as case 1.
- $g(2) + g(3) + g(4) + g(5) = 10$
- $a + 2b + 3c = 10$ and $a + b + c = 4$, so $b + 2c = 6$
- $(1, 0, 3)$ and $(0, 2, 2) \implies 4 + 6 = 10$
- If $g(5) = -1$, we have another 10 possibilities because the case is symmetric.
- If $g(1) = -1$, then $g(0) = g(2) = 3$.
- $g(3) + g(4) + g(5) = 7$
- $a + 2b + 3c = 7$ and $a + b + c = 3$
- $(1, 2, 0), (2, 0, 1) \implies 3 + 3 = 6$
- We have 6 more functions when $g(2), g(3), g(4) = -1$.
- $141 + 2 \cdot 10 + 4 \cdot 6 = \boxed{185}$.

Symmetry

- Some counting questions have a notion of symmetry
- Easy example: How many ways are there to arrange n people around a table if rotations are indistinguishable?
- $n!/n = (n-1)!$
- You can tell a question will involve symmetry if it includes words such as "indistinguishable" or involves something that can be spun/flipped like a cube or a circle
- We divided by n in the above example because all permutations belong into a group of equivalent rotations of size n. When beginning symmetry questions, it's important to first find these equivalence groups

Symmetry

2018 PUMaC C5

How many ways are there to color the 8 regions of a three-set Venn Diagram with 3 colors such that each color is used at least once? Two colorings are considered the same if one can be reached from the other by rotation and reflection.

- Replace 3 colors with n, and remove the restriction that we need each color at least once
- Note that there are 6 possible symmetries - 3 flips and 3 rotations. So, our answer is $n^8/6$
- Why is this wrong?
- Unlike our previous example, not every coloring is in "groups of 6." For example, if everything is colored the same color, there is only 1 symmetry, not 6. We can't just blindly divide by 6

- Note that 6 is the maximum possible number of elements in a "symmetry class," so all symmetry classes have $1, 2, 3$ or 6 elements. Let's go through them 1 by 1.
- Call the regions $R_\emptyset$, R_A, R_B, etc. Ignore $R_\emptyset$ and $R_{A,B,C}$
- How many symmetry classes have size 1?
- This means that no matter how it's rotated or flipped, it is still the same coloring. So, $R_A = R_B = R_C$ and $R_{A,B} = R_{A,C} = R_{B,C}$
- There are n^2 colorings with symmetry class size 1
- How many symmetry classes have size 2?
- Trick question! None of them do
- How many symmetry classes have size 3?
- This means that both $S_1 = \{R_A, R_B, R_C\}$ and $S_2 = \{R_{A,B}, R_{A,C}, R_{B,C}\}$ have at most 2 distinct elements
- We have 3 cases: $(|S_1|, |S_2|) = (1, 2), (2, 1), (2, 2)$

2018 PUMaC C5

- For $(1,2)$, choose a color for S_1, and choose 2 colors along with an orientation for S_2. So, we get $3n^2(n-1)$
- For $(2,2)$, we choose 2 colors for each S_1 and S_2, but note that we multiply by 3 once since they have the same orientation, so $3n^2(n-1)^2$
- In total, we have $3n^2(n-1)(n+1)$
- Finally, how many have symmetry class size 6?
- Just the remainder, so

$$n^6 - 3n^2(n^2-1) - n^2 = n^6 - 3n^4 + 2n^2$$

- So, our answer is

$$n^2\left(\frac{n^6-3n^4+2n^2}{6} + \frac{3n^2(n^2-1)}{3} + n^2\right) = \frac{n^8+3n^6+2n^4}{6}$$

2018 PUMaC C5

- Denote $f(n) = \frac{n^8+3n^6+2n^4}{6}$. How do we finish?
- We need at least one of each color, so by PIE our answer is

$$f(3) - 3f(2) + 3f(1) = 1485 - 240 + 3 = \boxed{1248}$$

Burnside's Lemma

- As we saw in the previous question, computations can get hard when there are too many symmetries
- We want a method to prevent casework on symmetries

Burnside's Lemma

Given a set X and a group of symmetries, G, let X^g be the number of elements in X fixed by $g \in G$. Then,

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

- Roughly speaking, this means that the total number of ways to count something is the average of the fixed sets over all symmetries
- We won't be proving this since it requires some nontrivial group theory

Classic

How many ways are there to color the faces of a cube with n colors if rotations are considered indistinguishable?

- Let's use Burnside's! We first need a list of all the symmetries
- This basically means: How many ways can we orient the cube? We have 6 ways to choose the top face and 4 ways to choose the front face. So, 24 total symmetries
- Let's try to characterize these symmetries more descriptively. Given a certain orientation, how can we move the cube?
  - The identity rotation: 1
  - 90^o rotation about a face: 6
  - 180^o rotation about a face: 3
  - 120^o rotations about a space diagonal: 8
  - 180^o rotations about a face diagonal: 6

Burnside's Lemma

- Let's go through the classes in order to find the # of fixed colorings
  - Identity:

Burnside's Lemma

- Let's go through the classes in order to find the # of fixed colorings
  - Identity: Everything is fixed under the identity, so n^6
  - 90^o face rotation:

Burnside's Lemma

- Let's go through the classes in order to find the # of fixed colorings
  - Identity: Everything is fixed under the identity, so n^6
  - 90^o face rotation: 2 faces are fixed, but the remaining 4 sides cycle by 1, so they all must be the same color, so n^3
  - 180^o face rotation:

Burnside's Lemma

- Let's go through the classes in order to find the # of fixed colorings
  - Identity: Everything is fixed under the identity, so n^6
  - 90^o face rotation: 2 faces are fixed, but the remaining 4 sides cycle by 1, so they all must be the same color, so n^3
  - 180^o face rotation: 2 faces are fixed and the remaining 4 sides swap in pairs of 2, so n^4
  - 120^o space diagonal:

Burnside's Lemma

- Let's go through the classes in order to find the # of fixed colorings
  - Identity: Everything is fixed under the identity, so n^6
  - 90^o face rotation: 2 faces are fixed, but the remaining 4 sides cycle by 1, so they all must be the same color, so n^3
  - 180^o face rotation: 2 faces are fixed and the remaining 4 sides swap in pairs of 2, so n^4
  - 120^o space diagonal: The faces are cycled in triplets, and the color within each triplet is the same, so n^2
  - 180^o face diagonal:

Burnside's Lemma

- Let's go through the classes in order to find the # of fixed colorings
  - Identity: Everything is fixed under the identity, so n^6
  - 90^o face rotation: 2 faces are fixed, but the remaining 4 sides cycle by 1, so they all must be the same color, so n^3
  - 180^o face rotation: 2 faces are fixed and the remaining 4 sides swap in pairs of 2, so n^4
  - 120^o space diagonal: The faces are cycled in triplets, and the color within each triplet is the same, so n^2
  - 180^o face diagonal: We form 3 pairs of swapped faces, so this gives n^3
- By Burnside's Lemma, our answer is

$$\frac{1}{24}\left(n^6 + 6n^3 + 3n^4 + 8n^2 + 6n^3\right) = \boxed{\frac{n^6 + 3n^4 + 12n^3 + 8n^2}{24}}$$