Lesson 11: Analytic Geometry

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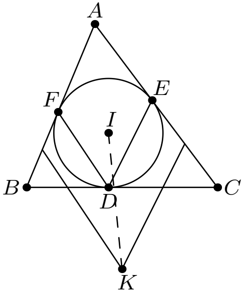
Problem of the Week

PotW

The incircle of triangle ABC touches BC, CA, AB at D, E, F respectively. Let I be its incenter, and K be the intersection of the B-midline of BDF and the C-midline of CDE. If the inradius is 10 and $AB + AC = 3BC = 294$, compute KI.
Note: the X-midline of triangle XYZ is defined to be the line passing through the midpoints of XY and XZ



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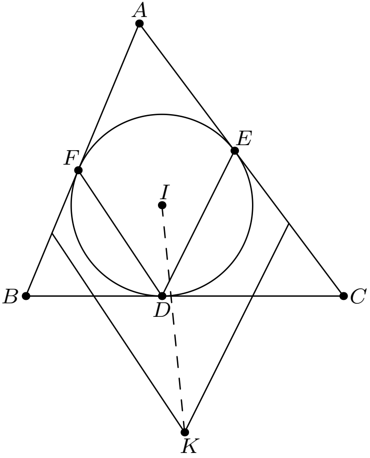
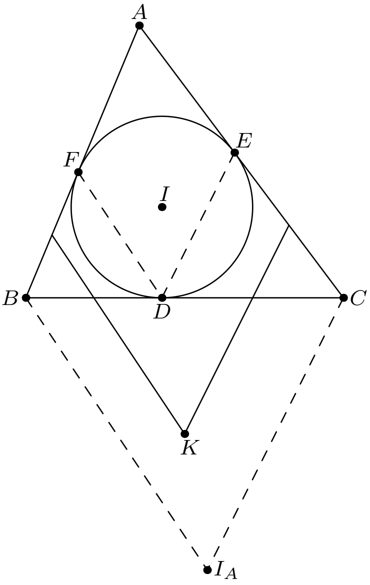


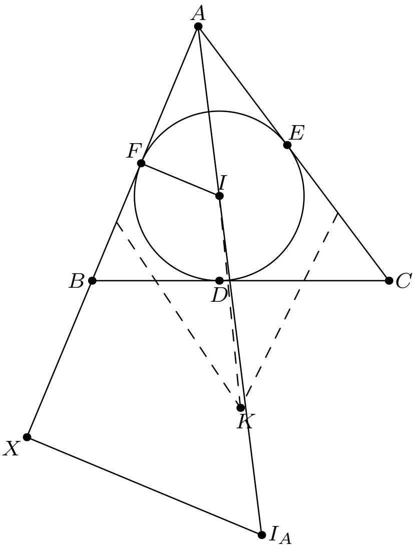
- What quantity can we use to compute KI?
- Its power with respect to the incircle is $KI^2 - 100$
- Does KI lie on any radical axes?
- Remember that points are just really small circles! What can we say about the radical axis of the incircle and point B?
- K is the radical center of the incircle and points B, C
- So, we just need to find $BK^2 = 49^2 + d^2$, where d is the distance between K and BC.

PotW



- We now know that the midlines are actually radical axes. Can we find any lines parallel to the B-midline?
- DF is, but there's one more not part of the problem statement
- The B-external angle bisector also is parallel.
- Where does the B midline hit line DI_A?
- Where does the C midline hit line DI_A?
- Both hit at the midpoint, so K is the midpoint of DI_A. Now, what is d?
- d is half of the A-exradius
- How can we compute the exradius given what we know?

PotW



- Note that $IFA \sim I_AXA$. Can we use this to compute the exradius?
- $AX = s$, $AF = s - a$. Now use similarity to find r_A.
- $r_A = \frac{s}{s-a} r$. Do we know all these values?
- Yes! $s = \frac{a+b+c}{2} = \frac{3a+a}{2} = 2a$, so this is just $2r$. Hence, $d = r$.
- Now, $BK^2 = 49^2 + r^2$, so $KI^2 - r^2 = 49^2 + r^2$
- $KI = \sqrt{49^2 + 200} = \boxed{51}$

Cartesian Coordinates

Many of you are already familiar with how to coordinate bash. Here are some tricks to minimize the amount of work you have to do:

- Identify from the beginning whether the question is feasible to coordinate bash. For example, we want to minimize number of angle conditions and non-linear intersections
- Choose a nice origin and axes. For example, if there is a point which is used extensively, set it as the origin, and if there are two prominent perpendicular lines, use them as axes
- Be smart when bashing! Before starting computation, find a way which guarantees the correct solution with minimal computation.
- Find synthetic observations at the beginning to save time. For example, proving a collinearity before beginning to bash can decrease computation time immensely

Coordinate Bashing Formulas

- Equation of a line: $y = mx + b$
- Distance Formula: $d^2 = (x - x_0)^2 + (y - y_0)^2$
- Parallel lines have equal slope, perpendicular lines have slopes which multiply to -1
- The slope of a line which hits the x-axis at an angle of θ is $\tan\theta$
- The distance from a point (x_0, y_0) to line $Ax + By + C = 0$ is

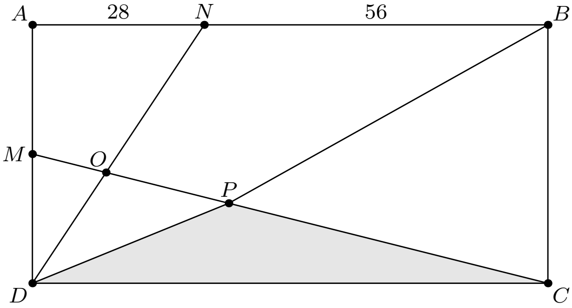
$$d = \left| \frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}} \right|$$

- **Shoelace Formula:** Suppose the polygon P has vertices (a_1, b_1), (a_2, b_2), ... , (a_n, b_n), listed in clockwise order. Then the area of P is

$$\frac{1}{2}\left|(a_1 b_2 + a_2 b_3 + \cdots + a_n b_1) - (b_1 a_2 + b_2 a_3 + \cdots + b_n a_1)\right|$$

2017 AIME II #10

Rectangle $ABCD$ has side lengths $AB = 84$ and $AD = 42$. Point M is the midpoint of $\overline{AD}$, point N is the trisection point of $\overline{AB}$ closer to A, and point O is the intersection of $\overline{CM}$ and $\overline{DN}$. Point P lies on the quadrilateral $BCON$, and $\overline{BP}$ bisects the area of $BCON$. Find the area of $\triangle CDP$.



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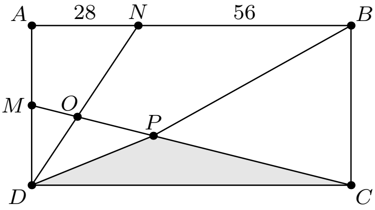

- What is a good choice of origin and axes?
- We set D to be the origin and DC, DA are the x, y axes. Which coordinates do we know immediately?
- $D = (0, 0)$, $C = (84, 0)$, $B = (84, 42)$, $A = (0, 42)$, $N = (28, 42)$, $M = (0, 21)$
- Now, let's compute the relevant lines.
- $DN : y = \frac{3}{2}x$, $MC : y = -\frac{1}{4}x + 21$
- What are the coordinates of O?

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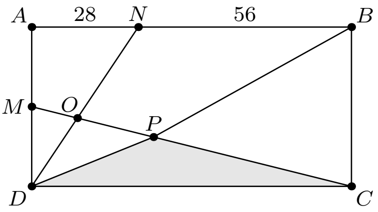

- Intersecting the lines, $\frac{3}{2}x = -\frac{1}{4}x + 21$, so $x = 12$ and $O = (12, 18)$
- Now, how can we compute the area of $BCON$?
- Note that $[BCON] = [BNDC] - [OCD] = \frac{56+84}{2} \cdot 42 - \frac{1}{2} \cdot 84 \cdot 18 = 2184$
- We need $[BPC] = 2184/2 = 1092$. How can we find the coordinates of P?
- $BC = 42$, so the height from P to BC is $\frac{2184}{42} = 52$. What is the x coordinate of P?

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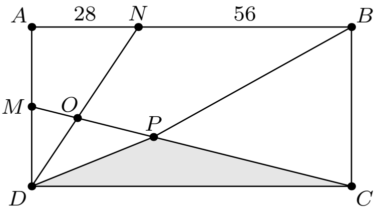
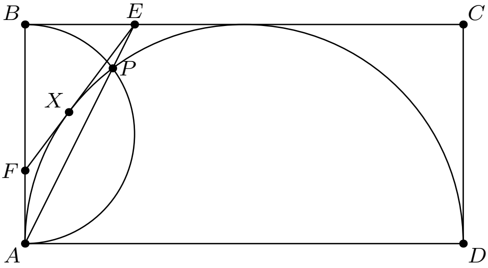


- P has x coordinate $84 - 52 = 32$, and since it lies on $y = -\frac{1}{4}x + 21$, $P = (32, 13)$
- How can we find the area of $[CPD]$?
- $[CPD] = \frac{1}{2} \cdot 84 \cdot 13 = \boxed{546}$

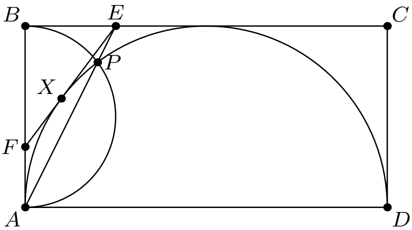
Cartesian Coordinates

2018 PUMaC

Consider rectangle $ABCD$ with $AB = 30$ and $BC = 60$. Construct circle T whose diameter is AD. Construct circle S whose diameter is AB. Let circles T and S intersect at P, so that $P \neq A$. Let AP intersect BC at E. Let F be the point on AB so that EF is tangent to the circle with diameter AD. Find the area of triangle AEF.

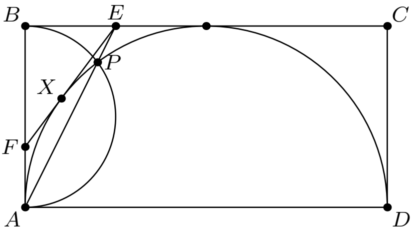


2018 PUMaC



- Set $A = (0, 0)$, $B = (0, 30)$, $C = (60, 30)$, $D = (60, 0)$.
- Equation of T is $(x - 30)^2 + y^2 = 900$.
- Equation of S is $x^2 + (y - 15)^2 = 225$.
- Subtracting, $60x - 30y = 0$, or $y = 2x$.
- Substituting this into the equation for T, $(x - 30)^2 + 4x^2 = 900$ or $x = 12$.
- $P = (12, 24)$, so the equation of AP is $y = 2x$. Coordinates of E?
- $E = (15, 30)$.

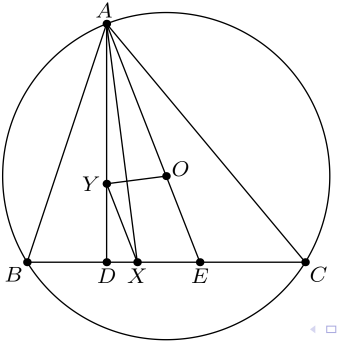
2018 PUMaC



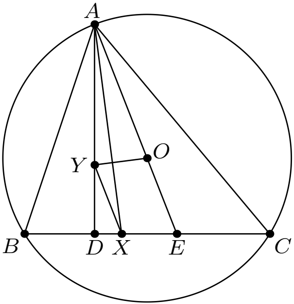
- Let X be the tangency point.
- What is EX?
- $EX = 15$, so X is on the circle $(x-15)^2 + (y-30)^2 = 225$.
- Remember the equation of T is $(x-30)^2 + y^2 = 900$.
- Subtracting, $30x - 60y + 900 = 0$ or $x = 2y - 30$.
- Thus, $(2y-60)^2 + y^2 = 900$ or $5y^2 - 240y + 2700 = 0$.
- We get $y = 18$, so $X = (6, 18)$.
- EX has equation $y = \frac{4}{3}x + 10$, so $F = (0, 10)$.
- $[AEF] = \frac{1}{2} \cdot 10 \cdot 15 = \boxed{75}$.

2014 HMMT

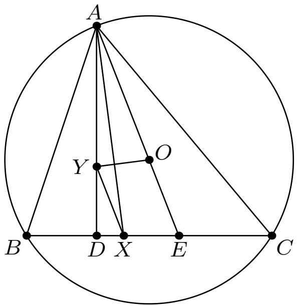
Let ABC be an acute triangle with circumcenter O such that $AB = 4$, $AC = 5$, $BC = 6$. Let D be the foot of the altitude from A to $\overline{BC}$ and E be the intersection of lines AO and BC. Suppose that X is on $\overline{BC}$ between D and E such that there is a point Y on $\overline{AD}$ satisfying $\overline{XY} \parallel \overline{AO}$ and $\overline{YO} \perp \overline{AX}$. Determine the length of BX.



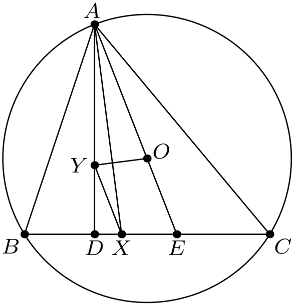
2014 HMMT



- Perpendicular and parallel lines: perfect for coordinates!
- Where should we put the origin? Think about which point makes the computations the easiest.
- Let's put D at the origin.
- Note that by Heron's formula, $[ABC] = \frac{15\sqrt{7}}{4}$, so $AD = \frac{2[ABC]}{BC} = \frac{5\sqrt{7}}{4}$.
- Thus, $A = (0, \frac{5\sqrt{7}}{4})$.
- Also, from the Pythagorean Theorem, $B = (-\frac{9}{4}, 0)$ and $C = (\frac{15}{4}, 0)$.
- Since O is on the perpendicular bisector of BC, the x-coordinate of O is $3 - \frac{9}{4} = \frac{3}{4}$.



- To find the y-coordinate, we can use the distance formula because we know $OB = R$.
- $R = \frac{abc}{4[ABC]} = \frac{8}{\sqrt{7}}$.
- The y-coordinate satisfies $3^2 + y^2 = \left(\frac{8}{\sqrt{7}}\right)^2$, or $y = \frac{\sqrt{7}}{7}$. So, $O = (\frac{3}{4}, \frac{\sqrt{7}}{7})$.
- $X = (a, 0)$. Let's use $\overline{XY} \parallel \overline{AO}$ to find Y.
- Slope of AO is $\frac{\frac{\sqrt{7}}{7} - \frac{5\sqrt{7}}{4}}{\frac{3}{4}} = -\frac{31\sqrt{7}}{21}$
- $Y = (0, \frac{31\sqrt{7}}{21}a)$
- To solve for a, we will use $\overline{YO} \perp \overline{AX}$.
- Slope of AX is $\frac{0 - \frac{5\sqrt{7}}{4}}{a - 0} = -\frac{5\sqrt{7}}{4a}$.



- Slope of OY is $\frac{\frac{\sqrt{7}}{7}-\frac{31\sqrt{7}}{21}a}{\frac{3}{4}-0} = \frac{4\sqrt{7}}{63}(3-31a)$.
- Product of the slopes is -1
- $-\frac{5\sqrt{7}}{4a} \cdot \frac{4\sqrt{7}}{63}(3-31a) = -1$
- This simplifies to $\frac{5}{9a}(3-31a) = 1$ or $15 - 155a = 9a$.
- Solving, $a = \frac{15}{164}$.
- Since $BD = \frac{9}{4}$, $BX = \frac{9}{4} + \frac{15}{164} = \boxed{\frac{96}{41}}$.

Complex Numbers

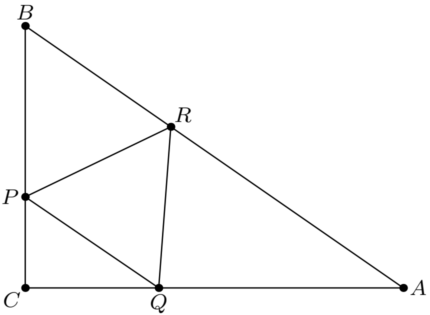
- Complex numbers are a useful alternative to Cartesian coordinates
- Instead of using an x-coordinate and y-coordinate, we use a real part an imaginary part
- This makes it easy to convert between complex numbers and Cartesian coordinates
- Complex numbers are commonly used for *rotations*
- If a point Z with coordinate z is rotated an angle θ counterclockwise about the origin, then its image is $(\cos\theta + i\sin\theta)z$
- Proof: when we multiply two complex numbers, their magnitudes multiply and their arguments add
- $|\cos\theta + i\sin\theta| = 1$, so the magnitude is the same but the argument is θ greater

Complex Numbers

- If a point Z with coordinate z is rotated an angle θ counterclockwise about a point A with coordinate a, then its image is $(\cos\theta + i\sin\theta)(z - a) + a$
- Proof: shift a to the origin: $a' = 0, z' = z - a$
- Rotate by θ: $(\cos\theta + i\sin\theta)z'$
- Shift a back to original position: $(\cos\theta + i\sin\theta)z' + a$
- So $(\cos\theta + i\sin\theta)(z - a) + a$
- Rotation can also be expressed in Cartesian coordinates
- The image of (x, y) under a clockwise rotation of θ around the origin is
$$(x\cos\theta - y\sin\theta, x\cos\theta + y\sin\theta)$$
- Proof: consider the product $(x + yi)(\cos\theta + i\sin\theta)$
- While rotation can be done entirely in Cartesian, complex numbers provide insight into why it works

2017 AIME I #15

The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}$, 5, and $\sqrt{37}$, as shown, is $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m+n+p$.



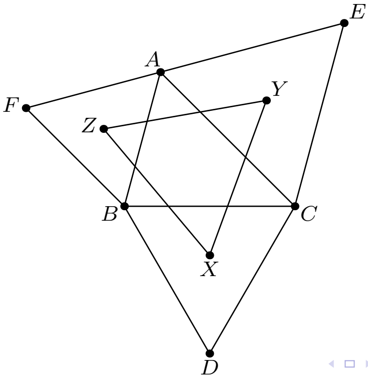
- Right triangle $\implies$ easy in Cartesian coordinates
- $C = (0, 0)$, $B = (0, 2\sqrt{3})$, $A = (5, 0)$
- Let $P = (0, p)$, $Q = (q, 0)$
- If PQR is equilateral, as shown, how do we compute R?
- Rotate Q 60° clockwise around P
- First translate P to the origin: $P' = (0, 0)$, $Q' = (q, -p)$
- Rotate Q' 60° clockwise around P': $\cos(60^\circ) = \frac{1}{2}$ and $\sin(60^\circ) = \frac{\sqrt{3}}{2}$
- $\left(\frac{1}{2}q + \frac{\sqrt{3}}{2}p, \frac{\sqrt{3}}{2}q - \frac{1}{2}p\right)$
- Shift P back to $(0, p)$; add p to y-coordinate
- $R = \left(\frac{1}{2}q + \frac{\sqrt{3}}{2}p, \frac{\sqrt{3}}{2}q + \frac{1}{2}p\right)$
- Need this to lie on $\overline{AB}$
- $\overline{AB}$ has equation $2\sqrt{3}x + 5y = 10\sqrt{3}$

2017 AIME I #15

- Plug in R:
- $2\sqrt{3}\left(\frac{1}{2}q + \frac{\sqrt{3}}{2}p\right) + 5\left(\frac{\sqrt{3}}{2}q + \frac{1}{2}p\right) = 10\sqrt{3}$
- Simplifies to $11p + 7\sqrt{3}q = 20\sqrt{3}$
- Now we want to express the area of triangle PQR
- $PQ^2 = p^2 + q^2$, $[PQR] = \frac{\sqrt{3}}{4}(p^2 + q^2)$
- To minimize $[PQR]$ we can minimize $p^2 + q^2$ subject to $11p + 7\sqrt{3}q = 20\sqrt{3}$
- Cleanest way is to use Cauchy-Schwarz:
- $(p^2 + q^2)(11^2 + (7\sqrt{3})^2) \geq (11p + 7\sqrt{3}q)^2 = 1200$
- $p^2 + q^2 \geq \frac{300}{67}$
- Equality is achievable at $p : q = 11 : 7\sqrt{3}$
- Area $\frac{\sqrt{3}}{4}(p^2 + q^2) = \frac{75\sqrt{3}}{67}$
- Answer $75 + 3 + 67 = \boxed{145}$

Napoleon's Theorem

Let ABC be a triangle. Points D, E, F satisfy that triangles BCD, CAE, and ABF are all equilateral and don't intersect the interior of triangle ABC. Show that the centers of these three equilateral triangles form an equilateral triangle.



Napoleon's Theorem

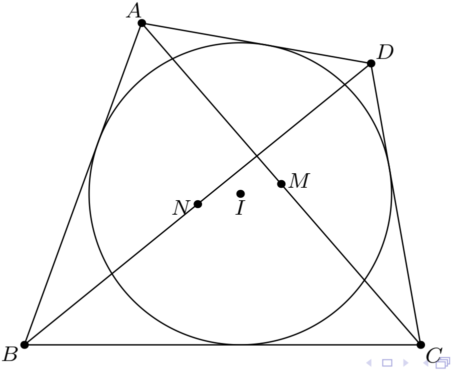
- We'll use complex numbers; a is the coordinate of A, etc.
- Let's try to express everything in terms of a, b, c
- D is the image of B under a 60° counterclockwise rotation around C
- $d = (\cos(60^\circ) + i\sin(60^\circ))(b - c) + c$
- $= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(b - c) + c$
- $= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) b + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) c$
- Now X is the centroid of triangle BCD so $x = \frac{b+c+d}{3}$
- $= \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) b + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) c$
- Similarly
  $y = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) c + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) a, \; z = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) a + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) b$
- (just cycle the variables: $x \mapsto y \mapsto z$ and $a \mapsto b \mapsto c$)

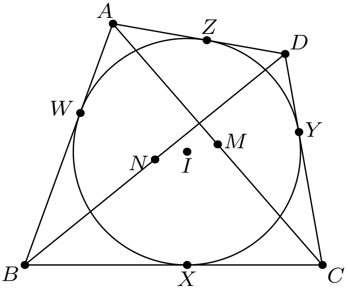
Napoleon's Theorem

- Now that we have the coordinates of X, Y, Z, how do we show triangle XYZ is equilateral?
- Can show that rotating Y 60° counterclockwise about X gives Z
- In coordinates this is $z = (\cos(60^\circ) + i\sin(60^\circ))(y - x) + x$
- Let's calculate the right-hand side
- $x = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) b + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) c, \; y = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) c + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) a$
- $y - x = \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) a - \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) b + \frac{\sqrt{3}}{3}ic$
- $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(y - x) = \left(\frac{1}{2} + \frac{\sqrt{3}}{3}i\right) a - \frac{\sqrt{3}}{3}ib + \left(-\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) c$
- $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(y - x) + x = \left(\frac{1}{2} + \frac{\sqrt{3}}{3}i\right) a + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) b$
- This is precisely z, so we're done

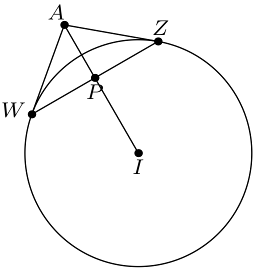
2017 HMMT Geometry # 10

Let $ABCD$ be a quadrilateral with an inscribed circle ω. Let I be the center of ω, and let $IA = 12$, $IB = 16$, $IC = 14$, and $ID = 11$. Let M be the midpoint of segment AC. Compute $\frac{IM}{IN}$, where N is the midpoint of segment BD.



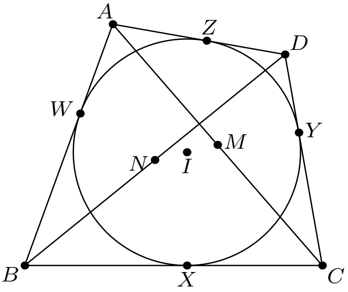


- Suppose the incircle touches the sides at W, X, Y, Z as shown
- Complex numbers are especially nice on the *unit circle*, the circle with radius 1 centered at the origin
- If z in on the unit circle (so $|z| = 1$ then $\overline{z} = \frac{1}{z}$
- It would be nice if the incircle was the unit circle
- Since we want to compute a ratio, it's okay if we shrink the incircle to the unit circle
- Then $IA = 12k$, $IB = 16k$, $IC = 14k$, $ID = 11k$ for some k



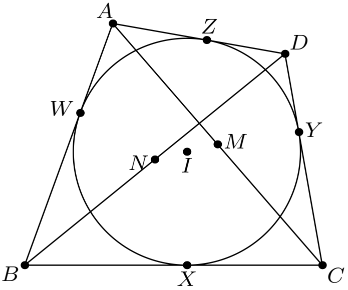
- We will use the following formula:
- *Claim.* $a = \frac{2zw}{z+w}$
- Let P be the midpoint of $\overline{WZ}$
- $\overline{API}$ perpendicular bisector of $\overline{WZ}$
- $\angle AWI = \angle AZI = 90^\circ$
- Similar triangles give $IP \cdot IA = IW^2 = 1$
- Since I, P, A collinear we have $\arg \overline{p} = 360^\circ - \arg p = 360^\circ - \arg a$
- So $\arg(\overline{p} \cdot a) = 360^\circ = 0^\circ$
- Then $\overline{p} \cdot a$ is a positive real
- But $|\overline{p} \cdot a| = |\overline{p}||a| = |p||a| = IP \cdot IA = 1$
- So $\overline{p} \cdot a = 1$

2017 HMMT Geometry #10



- $\overline{p} = \overline{\frac{z+w}{2}} = \frac{\overline{z}+\overline{w}}{2} = \frac{\frac{1}{z}+\frac{1}{w}}{2} = \frac{z+w}{2zw}$
- $a = \frac{1}{\overline{p}} = \frac{2zw}{z+w}$
- Similarly $b = \frac{2wx}{w+x}, \quad c = \frac{2xy}{x+y}, \quad d = \frac{2yz}{y+z}$
- $m = \frac{a+c}{2} = \frac{zw}{z+w} + \frac{xy}{x+y} = \frac{wxy+xyz+yzw+zwx}{(z+w)(x+y)}$
- $n = \frac{wxy+xyz+yzw+zwx}{(w+x)(y+z)}$
- I is the origin so $IM = |m|, \quad IN = |n|$
- $\frac{IM}{IN} = \frac{|m|}{|n|} = \left|\frac{m}{n}\right| = \left|\frac{(w+x)(y+z)}{(z+w)(x+y)}\right|$
- $= \frac{|w+x||y+z|}{|z+w||x+y|}$
- Now let's use the given lengths

2017 HMMT Geometry #10



- Since $IA = 12k$ we have $|a| = 12k$
- $|a| = \left|\frac{2zw}{z+w}\right| = \frac{2|z||w|}{|z+w|} = \frac{2}{|z+w|}$
- So $|z+w| = \frac{2}{12k}$
- Similarly $|w+x| = \frac{2}{16k}$, $|x+y| = \frac{2}{14k}$, $|y+z| = \frac{2}{11k}$
- Answer is

$$\frac{IM}{IN} = \frac{|w+x||y+z|}{|z+w||x+y|} = \frac{\frac{2}{16k}\cdot\frac{2}{11k}}{\frac{2}{12k}\cdot\frac{2}{14k}} = \boxed{\frac{21}{22}}$$