Lesson 10: Advanced Geometry

Adithya B., Brian L., William W., Daniel X.

August 2020

Problem of the Week

PotW

Tetrahedron $ABCD$ satisfies $AB = BC = CA$ and $DA = DB = DC$. Let E, F, G, H be the feet of the altitudes from A, B, C, D to their respective faces. If E, F, G, H all lie in the interiors of their respective faces, let the maximum value of $\frac{[EFGH]}{[ABCD]}$ be $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $m + n$.

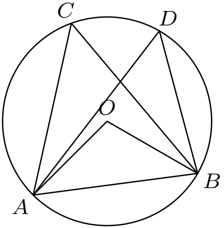
Problem of the Week

- We consider where the plane EFG intersects segments DA, DB, DC. Suppose these are X, Y, Z.
- Now, note that EFG and XYZ are both equilateral triangles, but EFG is the medial triangle of XYZ, so has $\frac{1}{4}$ the area.
- If the side length of ABC is 1, then let the side length of XYZ be k.
- Now, note that the ratio of the areas of the bases is $\frac{k^2}{4}$.
- Note that the ratio of the heights is $1 - k$. Thus, we wish to maximize $\frac{k^2(1-k)}{4}$.
- Now, note that we have $\frac{1}{3} = \frac{\frac{k}{2}+\frac{k}{2}+(1-k)}{3} \geq \sqrt[3]{\frac{k}{2} \cdot \frac{k}{2} \cdot (1-k)}$ by AM-GM
- Thus, we have that our expression is at most $\frac{1}{27}$, so our answer is $1 + 27 = \boxed{28}$.

Angle Chasing: Inscribed Angles

Theorem

Let A, B, and C be inscribed in a circle O. Then, $\angle AOB = 2\angle ACB$. *Also, angles that subtend the same arc are equal. That is, we have* $\angle ACB = \angle ADB$ *in the diagram below.*



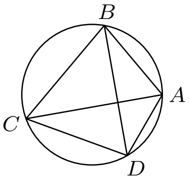
Proof.

Let $\angle ACO = \alpha$ and $\angle BCO = \beta$. Then, since $OA = OC$, $\angle OAC = \alpha$ and $\angle AOC = 180 - 2\alpha$. Similarly, $\angle BOC = 180 - 2\beta$. Therefore, we get, $\angle AOB = 360 - \angle AOC - \angle BOC = 2(\alpha + \beta) = 2\angle ACB$. □

Theorem

Let ABCD be a convex quadrilateral. Then, if the points A, B, C, D all lie on one circle, we say ABCD is cyclic. The following three statements are equivalent:

1. *ABCD is cyclic*
2. $\angle ABC + \angle CDA = 180$
3. $\angle ABD = \angle ACD$



Proof.

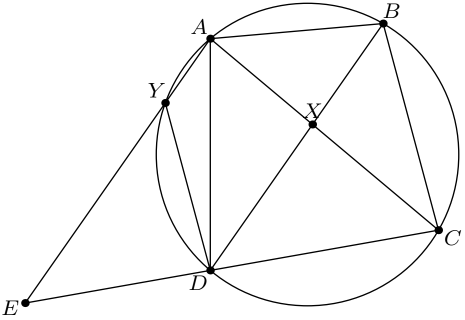
The forward direction follows from the Inscribed Angle Theorem.

□

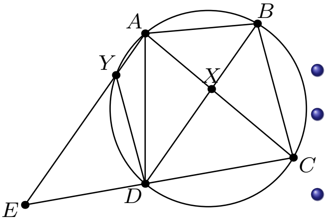
Angle Chasing

2016 PUMaC Geometry #7

Let $ABCD$ be a cyclic quadrilateral with circumcircle ω and let AC and BD intersect at X. Let the line through A parallel to BD intersect line CD at E and ω at $Y \neq A$. If $AB = 10, AD = 24, XA = 17$, and $XB = 21$, then the area of $\triangle DEY$ can be written in simplest form as $\frac{m}{n}$. Find $m + n$.



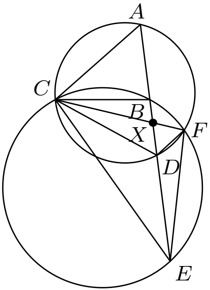
2016 PUMaC Geometry #7



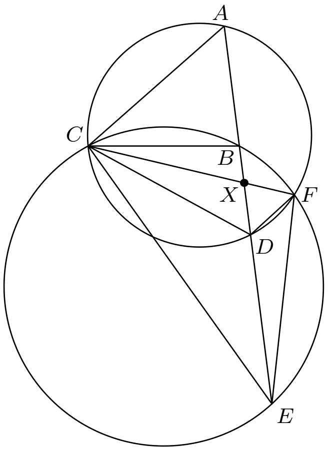
- First, note that we can find that $\angle DEY = \angle CDB = \angle XAB$ and $\angle AYD = \angle ABD$.
- Thus, we have that $DEY \sim XAB$.
- We also note that $ABDY$ is an isosceles trapezoid, $AY \parallel BD$
- This tells us that $DY = AB = 10$.
- Note that we are given all 3 sides of XAB, so we can find $[XAB] = 84$
- Now, note that we have that the ratio of the triangles DEY and XAB is $\frac{DY}{XB} = \frac{10}{21}$, so the ratio of their areas $\frac{[DEY]}{[XAB]} = \left(\frac{10}{21}\right)^2$
- Thus, we have that $[DEY] = 84\left(\frac{10}{21}\right)^2 = \frac{400}{21}$, so our answer is $400 + 21 = \boxed{421}$

2019 AIME I #13

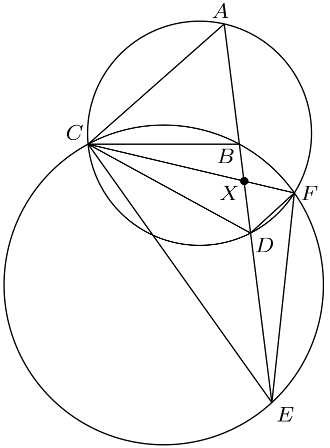
Triangle ABC has side lengths $AB = 4$, $BC = 5$, and $CA = 6$. Points D and E are on ray AB with $AB < AD < AE$. The point $F \neq C$ is a point of intersection of the circumcircles of $\triangle ACD$ and $\triangle EBC$ satisfying $DF = 2$ and $EF = 7$. Then BE can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.



2019 AIME I #13



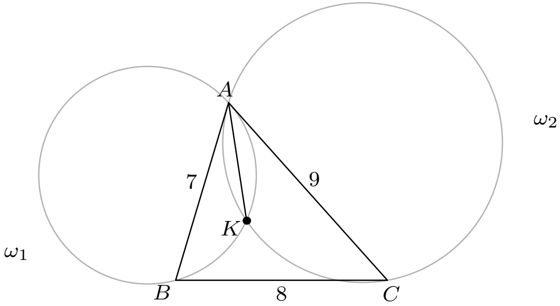
- Note that $\angle ACF = \angle ADF$ and $\angle BCF = \angle BEF$
- Subtracting these gives $\angle ACB = \angle DFE$
- We can calculate with Law of Cosines that $\cos \angle ACB = \frac{3}{4}$
- We can now use Law of Cosines on $\triangle DEF$ to find $DE = 4\sqrt{2}$.
- Now, let X be the intersection of CF and AE.
- Let $x = BX$ and $y = DX$.
- We have $FXD \sim AXC$ and $FXE \sim BXC$, so $FX = \frac{AX \cdot FD}{AC} = \frac{BX \cdot EF}{BC}$.

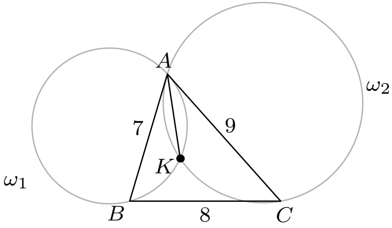


- Thus, we find that $\frac{x+4}{3} = \frac{7}{5}x$
- Thus, we may solve for x to get $x = \frac{5}{4}$
- Now, note that, by Power of a Point, we also have that $BX \cdot EX = CX \cdot FX = AX \cdot DX$, so we have $x(y + 4\sqrt{2}) = (x + 4)y$
- Thus, we find that $y = \sqrt{2}x = \frac{5\sqrt{2}}{4}$
- Now, we find $BE = x + y + 4\sqrt{2} = \frac{5+21\sqrt{2}}{4}$
- Thus, our answer is $5 + 21 + 2 + 4 = \boxed{32}$

2019 AIME II # 11

Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Then $AK = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



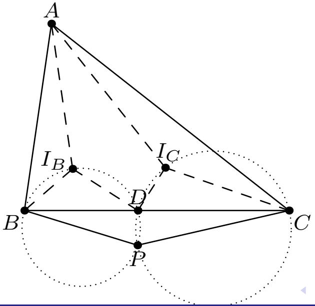


- The tangency condition yields $\angle ABK = \angle KAC$ and $\angle KAB = \angle ACK$.
- $\triangle BKA \sim \triangle AKC$.
- $\frac{AB}{AC} = \frac{7}{9}$
- Let $AK = x$. Then, $\frac{AK}{KC} = \frac{7}{9}$, so $KC = \frac{9}{7}x$.
- $\angle AKC = 180^\circ - \angle KAC - \angle KCA = 180^\circ - \angle KAC - \angle BAK = 180^\circ - \angle A$
- We can use the Law of Cosines on $\triangle AKC$.
- $\cos A = \frac{7^2+9^2-8^2}{2\cdot 7\cdot 9} = \frac{11}{21}$
- $\cos AKC = -\frac{11}{21}$
- $x^2 + \frac{81}{49}x^2 + 2x \cdot \frac{9}{7}x \cdot \frac{11}{21} = 81$
- $x = \frac{9}{2} \implies \boxed{011}$

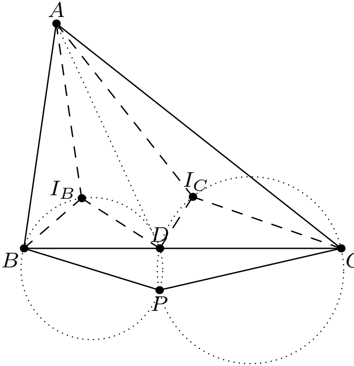
Angle Chasing

2009 AIME I # 15

In triangle ABC, $AB = 10$, $BC = 14$, and $CA = 16$. Let D be a point in the interior of $\overline{BC}$. Let I_B and I_C denote the incenters of triangles ABD and ACD, respectively. The circumcircles of triangles BI_BD and CI_CD meet at distinct points P and D. The maximum possible area of $\triangle BPC$ can be expressed in the form $a - b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.

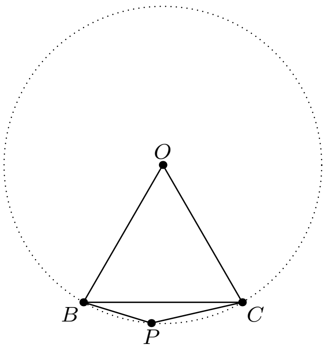


2009 AIME I # 15



- Let's look at $\angle BPC$.
- BI_BDP and CI_CDP are cyclic
- $\angle BPC = \angle BPD + CPD$
- $\angle BPC = 180^\circ - \angle BI_BD + 180^\circ - \angle CI_CD$
- $\angle BI_BD = 180^\circ - \angle DBI_B - \angle BDI_B = 180^\circ - \frac{1}{2}\angle B - \frac{1}{2}\angle ADB = 90^\circ + \frac{1}{2}\angle BAD$
- $\angle CI_CD = 90^\circ + \frac{1}{2}\angle CAD$
- $\angle BPC = 180^\circ - \frac{1}{2}\angle BAD - \frac{1}{2}\angle CAD = 180^\circ - \frac{1}{2}\angle A$.
- We can find $\angle A$ with the Law of Cosines.
- $\cos A = \frac{10^2+16^2-14^2}{2\cdot 10\cdot 16} = \frac{1}{2}$, so $\angle A = 60^\circ$ and $\angle BPC = 150^\circ$.
- $\angle BPC$ is fixed. What is the locus of P?

2009 AIME I # 15



- Arc of a circle! Let the center be O.
- What is $\angle BOC$?
- $\angle BOC = 360^\circ - 2 \cdot 150^\circ = 60^\circ$.
- Note $OB = OC$ and $\angle BOC = 60^\circ$, so BOC is equilateral.
- The radius is $OB = BC = 14$.
- Area is base times height.
- Maximum height is at the midpoint of arc BC.
- Maximum height is $14 - 7\sqrt{3}$.
- The area is $\frac{1}{2} \cdot 14 \cdot (14 - 7\sqrt{3}) = 98 - 49\sqrt{3}$.
- $98 + 49 + 3 = \boxed{150}$

Radical Axes

- Given a circle ω and a point P outside ω, let ℓ be any line through P
- Suppose ℓ meets ω at A and B; by Power of a Point $PA \cdot PB$ does not depend on ℓ
- If O is the center of ω and R is its radius, how else can we express this product?
- Choose $\ell = \overline{OP}$; what is $PA \cdot PB$ (in terms of OP and R)?
- Check that it's $OP^2 - R^2$
- If P is inside ω, then the answer is $R^2 - OP^2$
- We define $OP^2 - R^2$ to be the *power* of P with respect to ω
- Note that the power is negative for points inside the circle

Radical Axes

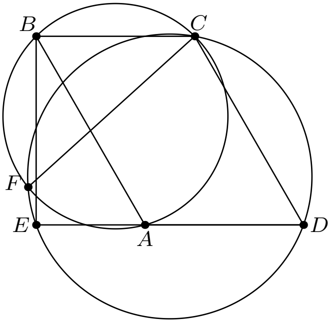
- Given two (nonconcentric) circles ω_1 and ω_2, what is the set of points with the same power with respect to both circles?
- The answer is a line, called the *radical axis* of ω_1 and ω_2
- This line is perpendicular to $\overline{O_1 O_2}$, where O_1 and O_2 are the centers of ω_1 and ω_2
- These results can be proved with a relatively simple use of coordinates, described in the handout
- Note that if two circles intersect, then their radical axis is the line through their intersection points

Radical Axis Theorem

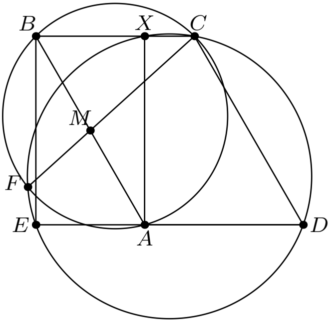
- One useful result about radical axes is the *Radical Axis Theorem*:
- Let ω_1, ω_2, ω_3 be pairwise nonconcentric circles. Let ℓ_1 be the radical axis of ω_2 and ω_3 and define ℓ_2 and ℓ_3 similarly. Then ℓ_1, ℓ_2, and ℓ_3 are either concurrent or all parallel.
- Proof: suppose ℓ_1 and ℓ_2 meet at P
- Since $P \in \ell_1$ P has equal powers with respect to ω_2 and ω_3
- Since $P \in \ell_2$ P has equal powers with respect to ω_1 and ω_3
- So P has equal powers with respect to ω_1 and ω_2
- Thus $P \in \ell_3$
- This shows that if any two radical axes intersect, then the three concur; thus they're concurrent or all parallel
- If they concur, we call the concurrency point the *radical center* of the three circles

2015 ARML Team # 10

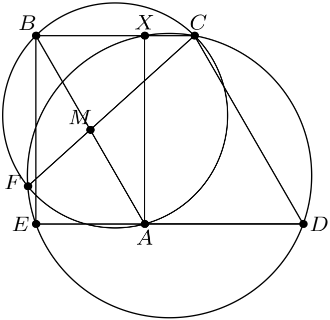
Let ABCD be a parallelogram with $m\angle A > 90^\circ$. Point E lies on $\overrightarrow{DA}$ such that $BE \perp AD$. The circumcircles of $\triangle ABC$ and $\triangle CDE$ intersect at points F and C. Given that $AD = 35$, $DC = 48$, and $CF = 50$, compute all possible values of AC.



2015 ARML Team # 10



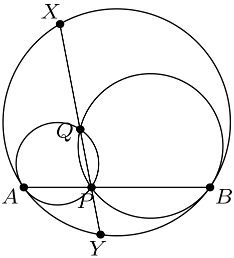
- CF is the radical axis of (ABC) and (CDE)
- Can we draw a third circle and use radical axis?
- Want this third circle to intersect (ABC) and (CDE) in nice points
- (ABE) works?
- Let X be foot from A to $\overline{BC}$
- $AEBX$ rectangle $\implies X \in (ABE)$
- $EX = AB = CD, \overline{CX} \parallel \overline{AD} \implies X \in (CDE)$
- Radical axis theorem on $(ABC), (CDE), (AEBX)$ $\implies \overline{CF}, \overline{AB}, \overline{EX}$ concur at midpoint of $\overline{AB}$



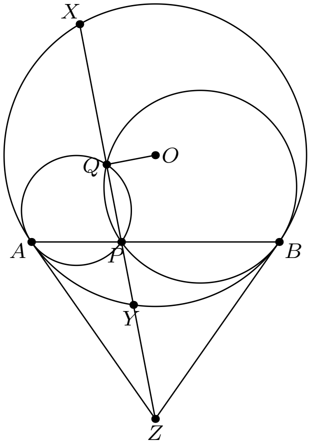
- Can now begin computation
- $AM = BM = 24, BC = 35, CF = 50$
- $AM \cdot BM = CM \cdot FM$
- $24 \cdot 24 = CM(50 - CM)$
- $CM = 18, 32$
- Median length formula on $\triangle ABC$
- $4CM^2 = 2AC^2 + 2BC^2 - AB^2$
- $AC^2 = \frac{4CM^2 - 2BC^2 + AB^2}{2}$
- $BC = 35, AB = 48, CM = 18, 32$
- $\boxed{5\sqrt{23}, 5\sqrt{79}}$

2019 AIME I # 15

Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

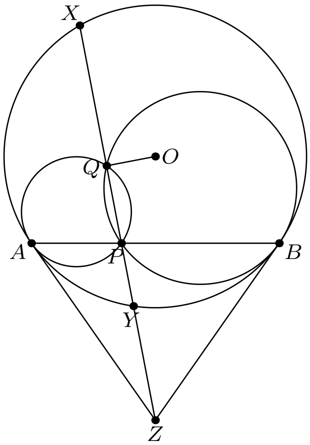


2019 AIME I # 15



- PQ radical axis of ω_1, ω_2
- Handling the tangencies: draw common tangent to ω, ω_1 at A and ω, ω_2 at B
- Let tangents meet at Z
- $ZA = ZB$ by equal tangents to ω
- ZA^2 power of Z wrt ω_1, ZB^2 power of Z wrt ω_2
- Z on radical axis $\overline{PQ}$
- Make another observation
- Is $AQBZ$ cyclic?
- $\angle ZQA = \angle PQA = \angle PAZ$ by tangency
- $\angle PAZ = \angle ABZ$ by isosceles $\triangle ZAB$
- So $\angle ZQA = \angle ZBA$, $AQBZ$ cyclic

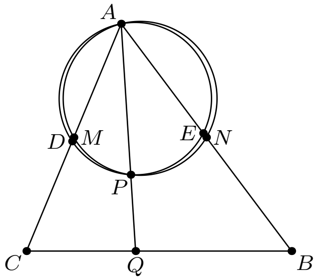
2019 AIME I # 15



- Let O be center of ω
- $\angle OAZ = \angle OBZ = 90^\circ$
- (ABZ) has diameter $\overline{OZ}$
- Z on $(ABZ) \implies \angle OQZ = 90^\circ$
- So Q midpoint of chord $\overline{XY}$
- $XQ = YQ = \frac{11}{2}, PY = \frac{11}{2} - PQ$
- $PX \cdot PY = PA \cdot PB$
- $\left(\frac{11}{2} + PQ\right)\left(\frac{11}{2} - PQ\right) = 3 \cdot 5$
- $PQ^2 = \frac{61}{4}$
- $\boxed{65}$

2010 AIME II #15

In triangle ABC, $AC = 13, BC = 14$, and $AB = 15$. Points M and D lie on AC with $AM = MC$ and $\angle ABD = \angle DBC$. Points N and E lie on AB with $AN = NB$ and $\angle ACE = \angle ECB$. Let P be the point, other than A, of intersection of the circumcircles of $\triangle AMN$ and $\triangle ADE$. Ray AP meets BC at Q. The ratio $\frac{BQ}{CQ}$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m - n$.



Linearity of Power

We begin by proving a lemma:

Linearity of Power

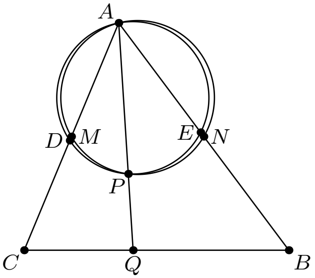
Given two circles ω_1, ω_2 in the plane, define

$$f(P) = \text{Pow}_{\omega_1}(P) - \text{Pow}_{\omega_2}(P)$$

Then, f is a linear function in P. Here, linear means that the function increments at a constant rate. So, $\frac{af(A)+bf(B)}{a+b} = f\left(\frac{a}{a+b}A + \frac{b}{a+b}B\right)$

- We use coordinates. Let $P = (x, y)$, and ω_1,ω_2 have radii r_1, r_2 and centers $(x_1, y_1), (x_2, y_2)$. What is $f(P)$?
- $f(P) = (x - x_1)^2 + (y - y_1)^2 - r_1^2 - (x - x_2)^2 - (y - y_2)^2 + r_2^2$.
- The quadratic terms cancel so f is linear!

2010 AIME II #15

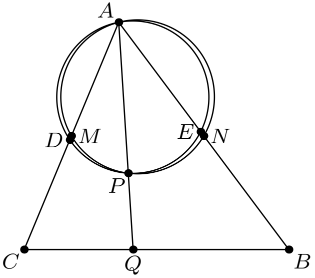


- How can we use linearity of power here?
- $f(P) = \text{Pow}_{(ADE)}(P) - \text{Pow}_{(AMN)}(P)$
- Q lies on the radical axis so $f(Q) = 0$
- How can we relate $\frac{BQ}{CQ}$ with $f(P)$?
- f is linear, so

$$\frac{QB \cdot f(C) + QC \cdot f(B)}{BC} = f(Q)$$

- $f(Q) = 0$ so $\frac{BQ}{CQ} = -\frac{f(B)}{f(C)}$
- Let's compute $f(B)$. What are BE, BN?
- $BN = \frac{AB}{2} = \frac{15}{2}$, $BE = \frac{14}{13+14} \cdot AB = \frac{70}{9}$
- So, $f(B) = BA \cdot BE - BA \cdot BN = 15\left(\frac{70}{9} - \frac{15}{2}\right) = \frac{25}{6}$

2010 AIME II #15



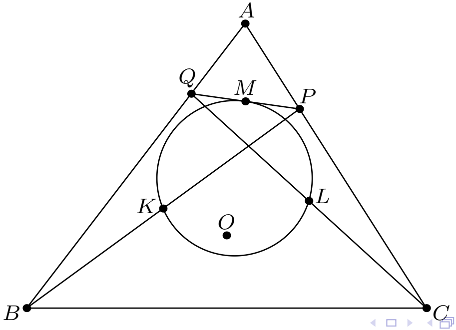
- Similarly, $CM = \frac{13}{2}$, $CD = \frac{14}{14+15} \cdot AC = \frac{182}{29}$
- $f(C) = AC \cdot (CD - CE) = 13\left(\frac{182}{29} - \frac{13}{2}\right) = -\frac{169}{58}$
- So, our final answer is

$$\frac{f(B)}{f(C)} = \frac{25/6}{169/58} = \frac{725}{507}$$

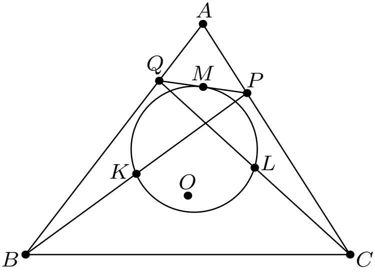
- $725 - 507 = \boxed{218}$

IMO 2009/2

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

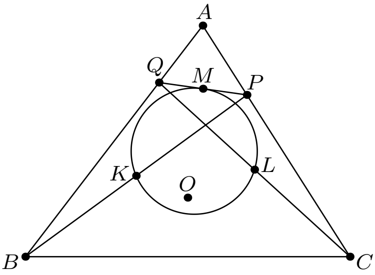


IMO 2009/2



- What does $OP = OQ$ mean?
- Since $\text{Pow}_{(ABC)}(X) = OX^2 - R^2$, it suffices to show that P, Q have the same power with respect to (ABC)
- How can we compute the powers of P, Q?
- We can use $AQ \cdot QB$ and $PC \cdot AP$
- Can we compute BQ, CP in terms of KM, ML, KL?
- KM is the midline of PQB so $QB = 2KM$. Similarly, $ML = 2PC$
- Now, try to use the tangent condition to get something about the angles

IMO 2009/2



- $KM \parallel AB$, so $\angle MLK = \angle KMQ = \angle AQP$. Similarly, $\angle MKL = \angle APQ$
- So, $APQ \sim MKL$. What does this tell you about AQ, AP?
- $\frac{AQ}{AP} = \frac{ML}{MK}$
- Putting it all together,

$$\frac{AQ \cdot QB}{AP \cdot PC} = \frac{ML}{MK} \cdot \frac{2MK}{2ML} = 1$$

- So, $AQ \cdot QB = AP \cdot PC$, as desired.