Trigonometry and Complex Numbers

Adithya B., Brian L., William W., Daniel X.

6/24

§1 Algebraic Trigonometry

When discussing algebraic trigonometry, the most useful identity is invariably the relations that are corollaries of the Pythagorean Theorem. For all angles θ, $\cos^2\theta + \sin^2\theta = 1$.

Another important concept is that $\cos(\theta) = \sin(90 - \theta)$, which is clear form the right triangle definitions of both functions.

Now, we will mention the trigonometric addition and subtraction formulas. We have the following relations for angles α and β:

Fact 1.1 (Addition and Subtraction Formulas)**.**

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha, \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}, \qquad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

Often times, problems will require you to use these formulas in the special case when $\alpha = \beta$. In these cases, it is usually faster to use the double-angle and half-angle formulas:

Fact 1.2 (Double Angle)**.**

$$\begin{aligned} \sin 2x &= 2\sin x\cos x, \\ \cos 2x &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x, \\ \tan 2x &= \frac{2\tan x}{1 - \tan^2 x}. \end{aligned}$$

Fact 1.3 (Half Angle)**.**

$$\begin{aligned} \sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}}, \\ \cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}}, \\ \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} &= \frac{1-\cos\theta}{\sin\theta}. \end{aligned}$$

Remark 1.4. Be careful with the plus or minus in Fact 1.3. If a problem requires you to use one of these formulas, there will usually be a condition on the angle restricting it to one of the values.

Example 1.5 (2014 AMC 12B #25)

Find the sum of all the positive solutions of

$$2\cos 2x\left(\cos 2x - \cos\left(\frac{2014\pi^2}{x}\right)\right) = \cos 4x - 1.$$

Solution. We see $\cos 2x$ multiple times on the left side, so this motivates us to write the right side as a function of $\cos 2x$ with the double angle identity.

$$2\cos 2x\left(\cos 2x - \cos\left(\frac{2014\pi^2}{x}\right)\right) = \cos 4x - 1 = 2\cos^2 2x - 2.$$

Now, we can divide by 2 and expand the left side.

$$\cos^2 2x - \cos 2x \cos\left(\frac{2014\pi^2}{x}\right) = \cos^2 2x - 1.$$

$$\cos 2x \cos\left(\frac{2014\pi^2}{x}\right) = 1.$$

Since $|\cos\theta| \le 1$, we must either have $\cos 2x = 1$ and $\cos\left(\frac{2014\pi^2}{x}\right) = 1$ or $\cos 2x = -1$ and $\cos\left(\frac{2014\pi^2}{x}\right) = -1$. Therefore, we can split into cases.

Case 1: $\cos 2x = 1$ and $\cos\left(\frac{2014\pi^2}{x}\right) = 1$:

This means that $2x = 2n\pi$ or $x = n\pi$ and $\frac{2014\pi^2}{x} = \frac{2014\pi}{n} = 2k\pi$ for integers n and k. This reduces to $nk = 1007$, so n can be any integer divisor of 1007, which would include 1, 19, 53, and 1007. Therefore. there are 4 solutions in this case and they have a sum of 1080π.

Case 2: $\cos 2x = -1$ and $\cos\left(\frac{2014\pi^2}{x}\right) = -1$:

This means that $2x = (2n+1)\pi$, or $x = \left(n + \frac{1}{2}\right)\pi$. Also,

$$\frac{2014\pi^2}{x} = \frac{2014\pi}{n+\frac{1}{2}} = \frac{4028\pi}{2n+1} = (2k+1)\pi$$

for integers n and k. However, this implies $(2n+1)(2k+1) = 4028$. which clearly has no solutions because both factors on the left side are odd integers and the right side is even.

Therefore, we can conclude that the sum of all solutions is $\boxed{1080\pi}$.

□

Example 1.6

Find the value of the sum

$$\sum_{k=1}^{\infty} \arctan\frac{1}{2k^2}.$$

Solution. To solve this problem, we really have to understand how adding arctangents works. Therefore, let us start with the hypothetical question of adding $\arctan x + \arctan y$.

Let $\arctan x + \arctan y = \theta$. Then, in an attempt to get rid of the inverse tangents on the left side, we can take a tangent of both sides.

$$\begin{aligned}\tan\theta &= \tan(\arctan x + \arctan y)\\ &= \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)}\\ &= \frac{x+y}{1-xy}\end{aligned}$$

But now, if we take the inverse tangent of both sides, we just get

$$\arctan x + \arctan y = \theta = \arctan\frac{x+y}{1-xy}.$$

So, adding two arctangents produces another arctangent. To understand our sum a little better, let's start computing partial sums using this identity for summing arctangents.

$$\sum_{k=1}^{1} \arctan\frac{1}{2k^2} = \arctan\frac{1}{2}.$$

$$\sum_{k=1}^{2} \arctan\frac{1}{2k^2} = \arctan\frac{1}{2} + \arctan\frac{1}{8} = \arctan\frac{\frac{1}{2}+\frac{1}{8}}{1-\frac{1}{2}\cdot\frac{1}{8}} = \arctan\frac{2}{3}.$$

$$\sum_{k=1}^{3} \arctan\frac{1}{2k^2} = \arctan\frac{2}{3} + \arctan\frac{1}{18} = \arctan\frac{\frac{2}{3}+\frac{1}{18}}{1-\frac{2}{3}\cdot\frac{1}{18}} = \arctan\frac{3}{4}.$$

It looks like we are starting to see a pattern, so we can form the following claim.

Claim 1.7 — For all positive integers n,

$$\sum_{k=1}^{n} \arctan\frac{1}{2k^2} = \arctan\frac{n}{n+1}.$$

Proof. We can prove this claim with induction. The base case $n = 1$ has already been shown above. Now assume that $\sum_{k=1}^{m} \arctan\frac{1}{2k^2} = \arctan\frac{m}{m+1}$. We will try to compute $\sum_{k=1}^{m+1} \arctan\frac{1}{2k^2}$. We have

$$\sum_{k=1}^{m+1} \arctan\frac{1}{2k^2} = \sum_{k=1}^{m} \arctan\frac{1}{2k^2} + \arctan\frac{1}{2(m+1)^2} = \arctan\frac{m}{m+1} + \arctan\frac{1}{2(m+1)^2}$$

by the inductive hypothesis. Now, we can use our arctangent addition formula to obtain

$$\begin{aligned}\sum_{k=1}^{m+1} \arctan\frac{1}{2k^2} = \arctan\frac{\frac{m}{m+1}+\frac{1}{2(m+1)^2}}{1-\frac{m}{m+1}\frac{1}{2(m+1)^2}} &= \arctan\frac{2m(m+1)^2+(m+1)}{2(m+1)^3-m}\\ &= \arctan\frac{(m+1)(2m^2+2m+1)}{(m+2)(2m^2+2m+1)}\\ &= \arctan\frac{m+1}{m+2}.\end{aligned}$$

This completes the induction. □

Now, we can easily evaluate our sum. As m approches infinity, we have

$$\lim_{m\to\infty} \frac{m}{m+1} = 1.$$

This means that

$$\sum_{k=1}^{\infty} \arctan \frac{1}{2k^2} = \arctan 1 = \frac{\pi}{4}.$$

$\square$

While the formulas above with some clever manipulation are usually sufficient to solve most algebraic trigonometry problems, another idea that can be useful is the method for converting the sum of trigonometric functions to a product and vice-versa.

Consider the sum $\sin(\alpha+\beta)+\sin(\alpha-\beta)$. By the addition and subtraction formulas, this is equal to $2\sin\alpha\cos\beta$. In other words, the sum of two sines can be written as a constant times the product of a sine and a cosine function. Thus, given the sum $\sin x + \sin y$, we can let $a = \frac{x+y}{2}$ and $b = \frac{x-y}{2}$ so that

$$\sin x + \sin y = \sin(a+b) + \sin(a-b) = 2\sin a \cos b = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).$$

This identity is known as a sum-to-product identity. We list the other sum-to-product identities in the fact below (which can all be derived with the same method).

Fact 1.8 (sum-to-product)**.**

$$\begin{aligned}
\cos x + \cos y &= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),\\
\cos x - \cos y &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right),\\
\sin x + \sin y &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),\\
\sin x - \sin y &= 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right).
\end{aligned}$$

It is usually not necessary to memorize any of these identities for problems because they can all be easily derived with the addition and subtraction formulas. Likewise, there are also product to sum identities:

Fact 1.9 (product-to-sum)**.**

$$\begin{aligned}
\cos\alpha\cos\beta &= \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)),\\
\sin\alpha\sin\beta &= \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)),\\
\sin\alpha\cos\beta &= \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta)).
\end{aligned}$$

That being said, we can now apply these concepts and identities in the following examples.

Example 1.10 (COMC)

Determine the sum of the angles A and B, where $0^\circ \leq A, B \leq 180^\circ$, and

$$\sin A + \sin B = \sqrt{\frac{3}{2}}, \qquad \cos A + \cos B = \sqrt{\frac{1}{2}}.$$

Solution. Using our sum-to-product identites, we have

$$2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = \sqrt{\frac{3}{2}}.$$

$$2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = \sqrt{\frac{1}{2}}.$$

As we have common terms, we can divide the equations. We obtain

$$\tan\left(\frac{A+B}{2}\right) = \sqrt{3}.$$

This means that $\frac{A+B}{2} = 60^\circ$ or $A + B = 120^\circ$. □

Example 1.11 (AIME)

Evaluate

$$\frac{\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \cdots + \cos 43^\circ + \cos 44^\circ}{\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots + \sin 43^\circ + \sin 44^\circ}.$$

Solution. Both of the sums seem hard to find directly, so motivated by the previous problem, we will try to use the sum-to-product formulas. Note the identity

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\alpha-\beta}{2}\right).$$

Therefore, when we apply this identity on the numerator, we want to produce common terms so that the resulting sum can be simplified. Therefore, we will pair the first term with the last term, the second term with the second-to-last term, and so on. This will produce a common term of $2\cos\left(\frac{45^\circ}{2}\right)$. We obtain

$$\begin{aligned}
\cos 1^\circ + \cos 2^\circ + \cdots + \cos 44^\circ &= (\cos 1^\circ + \cos 44^\circ) + (\cos 2^\circ + \cos 43^\circ) + \cdots \\
&= 2\cos\left(\frac{45^\circ}{2}\right)\cos\left(\frac{43^\circ}{2}\right) + 2\cos\left(\frac{45^\circ}{2}\right)\cos\left(\frac{41^\circ}{2}\right) + \cdots \\
&= 2\cos\left(\frac{45^\circ}{2}\right)\left(\cos\left(\frac{43^\circ}{2}\right) + \cos\left(\frac{41^\circ}{2}\right) + \cdots + \cos\left(\frac{1^\circ}{2}\right)\right).
\end{aligned}$$

If we try the same procedure on the denominator, we obtain the following:

$$\begin{aligned}
\sin 1^\circ + \sin 2^\circ + \cdots + \sin 44^\circ &= (\sin 1^\circ + \sin 44^\circ) + (\sin 2^\circ + \sin 43^\circ) + \cdots \\
&= 2\sin\left(\frac{45^\circ}{2}\right)\cos\left(\frac{43^\circ}{2}\right) + 2\sin\left(\frac{45^\circ}{2}\right)\cos\left(\frac{41^\circ}{2}\right) + \cdots \\
&= 2\sin\left(\frac{45^\circ}{2}\right)\left(\cos\left(\frac{43^\circ}{2}\right) + \cos\left(\frac{41^\circ}{2}\right) + \cdots + \cos\left(\frac{1^\circ}{2}\right)\right).
\end{aligned}$$

And now, when we divide, the second term cancels out:

$$\frac{\cos 1^\circ + \cos 2^\circ + \cdots + \cos 44^\circ}{\sin 1^\circ + \sin 2^\circ + \cdots + \sin 44^\circ} = \frac{2\cos\left(\frac{45^\circ}{2}\right)\left(\cos\left(\frac{43^\circ}{2}\right) + \cos\left(\frac{41^\circ}{2}\right) + \cdots + \cos\left(\frac{1^\circ}{2}\right)\right)}{2\sin\left(\frac{45^\circ}{2}\right)\left(\cos\left(\frac{43^\circ}{2}\right) + \cos\left(\frac{41^\circ}{2}\right) + \cdots + \cos\left(\frac{1^\circ}{2}\right)\right)}$$
$$= \cot\left(\frac{45^\circ}{2}\right).$$

Now, this value might seem hard to find, but we can just use one of our half-angle formulas:

$$\cot\left(\frac{45^\circ}{2}\right) = \frac{1}{\tan\left(\frac{45^\circ}{2}\right)} = \frac{1 + \cos 45^\circ}{\sin 45^\circ} = \frac{1 + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \sqrt{2} + 1.$$

□

§2 Complex Numbers

Recall that a *complex* number is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. We call a the *real part* and b the *imaginary part* of $a + bi$. If $z = a + bi$, then $a - bi$ is called the *conjugate* of z and is denoted $\overline{z}$.

Adding and subtracting complex numbers is simple: we just add the real parts and the imaginary parts. To multiply, we use the distributive property. To divide, it is necessary to multiply the numerator and the denominator of the complex number by the conjugate of the denominator, which will turn the denominator into a real number. We demonstrate one example of each operation here:

$$(3 + 4i) + (1 + 2i) = (3 + 4) + (1 + 2)i = 4 + 6i,$$
$$(3 + 4i) - (1 + 2i) = (3 - 1) + (4 - 2)i = 2 + 2i,$$
$$(3 + 4i)(1 + 2i) = 3(1 + 2i) + 4i(1 + 2i) = 3 + 6i + 4i + 8i^2 = -5 + 10i,$$
$$\frac{3 + 4i}{1 + 2i} = \frac{(3 + 4i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{11 - 2i}{5}.$$

Many problems involving complex numbers are approachable in the same way as typical problems involving real numbers. The reason is that the algebra of complex numbers is no different from the algebra of real numbers: one can perform the four basic operations, solve linear equations, use the quadratic formula, etc. in exactly the same way as real numbers. For this reason, we'll focus on problems that involve special properties unique to complex numbers.

One useful fact is the following:

Theorem 2.1

If a polynomial f has real coefficients and $f(z) = 0$ for some complex number z, then $f(\overline{z})$ as well.

Proof. This is a consequence of an useful property of complex conjugation: it behaves well with respect to basically every operation. That is,

$$\overline{a + b} = \overline{a} + \overline{b},\ \overline{a - b} = \overline{a} - \overline{b},\ \overline{ab} = \overline{a}\overline{b},\ \overline{\frac{a}{b}} = \frac{\overline{a}}{\overline{b}}.$$

If our polynomial is $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $a_i = \overline{a_i}$ for any a_i because a_i is real, so

$$\begin{aligned} f(\overline{z}) &= a_n\overline{z}^n + a_{n-1}\overline{z}^{n-1} + \cdots + a_0 \\ &= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \cdots + a_0 \\ &= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_0} \\ &= \overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0} \\ &= \overline{f(z)} \\ &= 0. \end{aligned}$$

□

Let's see this fact in action:

Example 2.2 (1995 AIME # 5)

For certain real values of a, b, c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$. Find b.

Solution. Suppose we have two roots p, q whose product is $13 + i$. Since our given polynomial has real coefficients, we know that $\overline{p}$ and $\overline{q}$ are also roots of the quartic. But p and q can't be conjugates of each other, or else pq would be real. This implies that $p, q, \overline{p}, \overline{q}$ are the four distinct roots of the quartic. The other equation we're given is

$$\overline{p} + \overline{q} = 3 + 4i.$$

By properties of the conjugate, we see that we also have

$$p + q = 3 - 4i, \ \overline{pq} = 13 - i.$$

We now seek b; by Vieta this is equal to

$$pq + p\overline{p} + q\overline{q} + p\overline{q} + q\overline{p} + \overline{pq} = pq + \overline{pq} + (p+q)(\overline{p} + \overline{q}).$$

We can now plug in the values we know, for an answer of

$$13 + i + (3 + 4i)(3 - 4i) + 13 - i = \boxed{51}.$$

□

We'll now focus on the relationship between trigonometry and complex numbers. So far, we've defined a complex number with two real numbers: the real part and the imaginary part. However, we can define complex numbers in another manner.

Definition 2.3. The *magnitude* or *absolute value* of a complex number, denoted $|z|$, is its distance from the origin on the complex plane. Thus $|a + bi| = \sqrt{a^2 + b^2}$.

Definition 2.4. The *argument* of a complex number, denoted $\arg(z)$, is the angle between the line connecting the origin and z in the complex plane and the positive x-axis. Thus

$$\arg(a + bi) = \begin{cases} \arctan\left(\frac{b}{a}\right) & a + bi \text{ in quadrants I or IV} \\ \arctan\left(\frac{b}{a}\right) + 180^\circ & \text{in quadrants II or III} \end{cases}.$$

For example, $\arg(i) = 90^\circ$ and $\arg\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = 225^\circ$.

Thus a complex number z can be specified with two numbers, its magnitude r and its argument θ. We may write

$$z = r(\cos\theta + i\sin\theta).$$

Complex numbers have a lot of structure when it comes to multiplication. We state the following result without proof:

Theorem 2.5

For any complex numbers a, b,

$$|ab| = |a||b|, \ \arg(ab) = \arg(a) + \arg(b).$$

A corollary of this fact is the following:

Theorem 2.6 (De Moivre's Formula)

For any angle θ and positive integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Proof. Let's induct on n. The base case $n = 1$ is direct; for the inductive step suppose that we have the equality

$$(\cos\theta + i\sin\theta)^{n-1} = \cos(n-1)\theta + i\sin(n-1)\theta.$$

Multiplying both sides by $\cos\theta + i\sin\theta$, we have

$$(\cos\theta + i\sin\theta)^n = (\cos(n-1)\theta + i\sin(n-1)\theta)(\cos\theta + i\sin\theta).$$

But expanding, the RHS is

$$\begin{aligned}&(\cos(n-1)\theta + i\sin(n-1)\theta)(\cos\theta + i\sin\theta)\\ &= (\cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta) + (\cos(n-1)\theta\sin\theta + \sin(n-1)\theta\cos\theta)i\\ &= \cos n\theta + i\sin n\theta.\end{aligned}$$

Thus

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

and our induction is complete. □

Corollary 2.7

The n solutions to the equation $z^n = 1$ are

$$\cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

for $0 \le k \le n-1$. We call these n complex numbers the *nth roots of unity.*

If you misremember De Movire's, you might not get the wrong answer:

Example 2.8 (2005 AIME II # 9)
For how many positive integers n less than or equal to 1000 is

$$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$

true for all real t?

Solution. This looks like De Moivre, but not quite. To make it De Moivre, let's rewrite the LHS:

$$(\sin t + i\cos t)^n = (\cos(90 - t) + i\sin(90 - t))^n.$$

Now we can use De Moivre: this is equal to

$$\cos(90 - t)n + i\sin(90 - t)n.$$

Comparing this to $\sin nt + i\cos nt$, we need the two equations

$$\cos(90 - t)n = \sin nt, \ \sin(90 - t)n = \cos nt$$

to always be true. It's helpful to rewrite them as

$$\cos(90n - nt) = \sin nt, \ \sin(90n - nt) = \cos nt$$

because we can now write $x = nt$, and the two equations

$$\cos(90n - x) = \sin x, \ \sin(90n - x) = \cos x$$

must always be true. Now it's easy to see that this occurs whenever $n \equiv 1 \pmod 4$ (say by expanding $\cos(90n - x)$ to get $\cos 90n \cos x + \sin 90n \sin x = \sin x$ so we must always have $\cos 90n = 0$ and $\sin 90n = 1$, and similarly for the second equation). Thus the answer is $\boxed{250}$. □

Example 2.9 (2000 AIME II #9)
Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3°$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

Solution. Let's solve for z with the quadratic formula:

$$z + \frac{1}{z} = 2\cos 3° \implies z^2 - 2\cos 3° z + 1 = 0 \implies z = \frac{2\cos 3° \pm \sqrt{(2\cos 3°)^2 - 4}}{2}$$

which simplifies to

$$\cos 3° \pm \sqrt{(\cos 3°)^2 - 1} = \cos 3° \pm i\sin 3°$$

Since z is in this form, we can use De Moivre!

$$z^{2000} = \cos 6000° \pm i\sin 6000° = \cos 240° \pm i\sin 240° = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

Regardless of which value of z^{2000} we pick, we find that $z^{2000} + \frac{1}{z^{2000}} = -1$. So the answer is $\boxed{0}$. □

We can even use the arguments of complex numbers to model what seem to be completely trigonometric problems:

Example 2.10 (2008 AIME I # 8)

Find the positive integer n such that

$$\arctan\frac{1}{3}+\arctan\frac{1}{4}+\arctan\frac{1}{5}+\arctan\frac{1}{n}=\frac{\pi}{4}.$$

Solution. This problem can be solved with the same method used in §1 for adding inverse tangents, but we will illustrate the complex number method here. The tangent of the argument of a complex number is particularly easy to represent: it's the ratio of the imaginary part to the real part. For example,

$$\tan(\arg(3+2i))=\frac{2}{3}.$$

How do we get a complex number whose argument is $\arctan\frac{1}{3}+\arctan\frac{1}{4}$? Since arguments add when we multiply two complex numbers, we just need to find two complex numbers whose arguments are $\arctan\frac{1}{3}$ and $\arctan\frac{1}{4}$. These are easy to find; choose $3+i$ and $4+i$, respectively. Thus we have

$$\arg((3+i)(4+i))=\arctan\frac{1}{3}+\arctan\frac{1}{4}$$

By the same reasoning, we see that

$$\arg((3+i)(4+i)(5+i)(n+i))=\arctan\frac{1}{3}+\arctan\frac{1}{4}+\arctan\frac{1}{5}+\arctan\frac{1}{n}=\frac{\pi}{4}.$$

Let's expand the inside: we compute that

$$(3+i)(4+i)(5+i)(n+i)=(48n-46)+(46n+48)i.$$

We know that the argument of this complex number is $\frac{\pi}{4}$; this means that the real part is equal to the imaginary part. So

$$48n-46=46n+48 \implies n=\boxed{47}.$$

□

Remark 2.11. It's also possible to solve this problem without complex numbers, using the tangent addition formula. Try it! (Take the tangents of both sides of the given equation.)

§3 Problems

Problem 3.1 (2013 AMC 12B #25)**.** Let G be the set of polynomials of the form

$$P(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_2z^2 + c_1z + 50,$$

where $c_1, c_2, \cdots, c_{n-1}$ are integers and $P(z)$ has n distinct roots of the form $a + ib$ with a and b integers. How many polynomials are in G?

Problem 3.2 (2012 AIME I #6)**.** The complex numbers z and w satisfy $z^{13} = w$, $w^{11} = z$, and the imaginary part of z is $\sin\left(\frac{m\pi}{n}\right)$ for relatively prime positive integers m and n with $m < n$. Find n.

Problem 3.3 (2018 AIME I #6)**.** Let N be the number of complex numbers z with the properties that $|z| = 1$ and $z^{6!} - z^{5!}$ is a real number. Find the remainder when N is divided by 1000.

Problem 3.4 (AoPS)**.** Evalute the product

$$(\sin 1^\circ)(\sin 3^\circ)(\sin 5^\circ)\cdots(\sin 177^\circ)(\sin 179^\circ).$$

Problem 3.5 (1984 AIME #13)**.** Find the value of $10\cot(\cot^{-1} 3 + \cot^{-1} 7 + \cot^{-1} 13 + \cot^{-1} 21)$.

Problem 3.6 (NYSML)**.** If $\sin\theta + \cos\theta + \tan\theta + \cot\theta + \sec\theta + \csc\theta = 7$, then find the value of $\sin 2\theta$.

Problem 3.7 (2020 CMIMC Algebra #4)**.** For all real numbers x, let $P(x) = 16x^3 - 21x$. What is the sum of all possible values of $\tan^2\theta$, given that θ is an angle satisfying

$$P(\sin\theta) = P(\cos\theta)?$$

Problem 3.8 (2006 AIME I #12)**.** Find the sum of the values of x such that $\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x$, where x is measured in degrees and $100 < x < 200$.

Problem 3.9 (2014 AIME II #13)**.** The polynomial

$$P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$$

has 34 complex roots of the form $z_k = r_k[\cos(2\pi a_k) + i\sin(2\pi a_k)], k = 1, 2, 3, \ldots, 34$, with $0 < a_1 \le a_2 \le a_3 \le \cdots \le a_{34} < 1$ and $r_k > 0$. Given that $a_1 + a_2 + a_3 + a_4 + a_5 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

Problem 3.10 (2018 PUMaC Live Round)**.** Find x^2 given that $\tan^{-1} x + \tan^{-1} 3x = \frac{\pi}{6}$ and $0 < x < \frac{\pi}{6}$.

Problem 3.11 (2000 AIME II #15)**.** Find the least positive integer n such that $\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}$.

Problem 3.12 (2018 Lehigh Contest #40)**.** What is the largest value of $\sin x$ which satisfies

$$\sin x + \sin 2x + \cos x = 0?$$

Problem 3.13 (2003 AIME I #11)**.** An angle x is chosen at random from the interval $0^\circ < x < 90^\circ$. Let p be the probability that the numbers $\sin^2 x, \cos^2 x$, and $\sin x \cos x$ are not the lengths of the sides of a triangle. Given that $p = d/n$, where d is the number of degrees in $\arctan m$ and m and n are positive integers with $m + n < 1000$, find $m + n$.

Problem 3.14 (2019 PUMaC Live Round)**.** Compute

$$\left\lfloor \sum_{n=0}^{49} \sin\left(\frac{\pi n}{100}\right) \right\rfloor.$$

Problem 3.15 (2001 AIME II #14)**.** There are $2n$ complex numbers that satisfy both $z^{28} - z^8 - 1 = 0$ and $|z| = 1$. These numbers have the form $z_m = \cos\theta_m + i\sin\theta_m$, where $0 \le \theta_1 < \theta_2 < \cdots < \theta_{2n} < 360$ and angles are measured in degrees. Find the value of $\theta_2 + \theta_4 + \cdots + \theta_{2n}$.

Problem 3.16 (2009 AIME II #14)**.** The sequence (a_n) satisfies $a_0 = 0$ and $a_{n+1} = \frac{8}{5}a_n + \frac{6}{5}\sqrt{4^n - a_n^2}$ for $n \ge 0$. Find the greatest integer less than or equal to a_{10}.

Problem 3.17 (2020 Lehigh Contest #36)**.** Let x and y be positive real numbers and θ be an angle which is not an integer multiple of $\pi/2$. Suppose

$$\frac{\sin\theta}{x} = \frac{\cos\theta}{y} \text{ and } \frac{\cos^4\theta}{x^4} + \frac{\sin^4\theta}{y^4} = \frac{7\sin 2\theta}{x^3y + xy^3}.$$

What is the value of $\frac{x}{y} + \frac{y}{x}$?

Problem 3.18 (2018 CMIMC Algebra #7)**.** Compute

$$\sum_{k=0}^{2017} \frac{5 + \cos\left(\frac{\pi k}{1009}\right)}{26 + 10\cos\left(\frac{\pi k}{1009}\right)}.$$

Problem 3.19 (Classical)**.** Given any seven real numbers, prove there are two of them, x and y, such that

$$0 \le \frac{x - y}{1 + xy} \le \frac{1}{\sqrt{3}}.$$

Problem 3.20 (TSTST 2017/3)**.** Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)},$$

where f and g are polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.