Summations B

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§1 Combinatorial Arguments

When faced with a sum involving binomial coefficients, one approach is to consider it *combinatorially*. That is, we try to relate the given sum to the number of possibilities for a certain combinatorial structure. Typically, we create our own combinatorial situation, and proceed to count the number of ways to do something using two different methods. Then the two answers we get from these two different methods must be equal. This is best explained by example, so we will demonstrate it on a famous and useful identity.

Example 1.1 (Vandermonde's Identity)

For nonnegative integers m, n, r,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

Solution. Let's try to interpret both sides of the equation combinatorially. The left-hand side is very simple: it is the number of ways to choose r objects from $m + n$ objects. But the right hand side needs some work. How can we interpret it as the number of ways to do something?

We note that $\binom{m}{k}$ is the number of ways to choose k objects from m objects, while $\binom{n}{r-k}$ is the number of ways to choose $r - k$ objects from n objects. For example, $\binom{m}{k}$ is the number of ways to pick k red stones from a pile of m red stones, and $\binom{n}{r-k}$ is the number of ways to pick $r - k$ blue stones from a pile of n blue stones. Thus $\binom{m}{k}\binom{n}{r-k}$ is the number of ways to pick k red stones and $r - k$ blue stones from m red stones and n blue stones.

Now how do we interpret the sum of $\binom{m}{k}\binom{n}{r-k}$ from $k = 0$ to $k = r$? This corresponds to the number of ways to choose r stones of *any* color from a pile of m red stones and n blue stones. But there are clearly $\binom{m+n}{k}$ ways to do this. Thus we have equated

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

because both sides of the equation are equal to the number of ways to choose r stones from a pile of m red stones and n blue stones. □

Vandermonde's Identity can be fairly useful in computational contests as a way to simplify combinatorial sums. Let's see it in action with a problem from a recent AIME:

Example 1.2 (2017 AIME I # 7)

For nonnegative integers a and b with $a+b \leq 6$, let $T(a,b) = \binom{6}{a}\binom{6}{b}\binom{6}{a+b}$. Let S denote the sum of all $T(a,b)$, where a and b are nonnegative integers with $a+b \leq 6$. Find the remainder when S is divided by 1000.

Solution. It's important to remember our basic algebraic methods for dealing with sums. We're asked to compute the double sum

$$\sum_{0 \leq a,b \leq 6} \binom{6}{a}\binom{6}{b}\binom{6}{a+b}.$$

Let's sum across the values of $a+b$; that is, we can rewrite the sum as

$$\sum_{n=0}^{6} \sum_{\substack{a+b=n \\ a,b \geq 0}} \binom{6}{a}\binom{6}{b}\binom{6}{a+b} = \sum_{n=0}^{6} \binom{6}{n} \sum_{\substack{a+b=n \\ a,b \geq 0}} \binom{6}{a}\binom{6}{b}$$

because we're just choosing the order in which we will calculate the sum. By Vandermonde, the inner sum is

$$\sum_{\substack{a+b=n \\ a,b \geq 0}} \binom{6}{a}\binom{6}{b} = \sum_{a=0}^{n} \binom{6}{a}\binom{6}{n-a} = \binom{12}{n}.$$

Now the entire sum is

$$\sum_{n=0}^{6} \binom{6}{n}\binom{12}{n}.$$

This doesn't look quite like Vandermonde yet. We need the sum of the bottoms of the two binomial coefficients to remain constant. To do this, we use a little sleight of hand and write the sum as

$$\sum_{n=0}^{6} \binom{6}{6-n}\binom{12}{n}.$$

Now we can use Vandermonde! The sums evaluates to $\binom{18}{6}$, and we leave as an exercise to the reader to check that the final answer is $\boxed{564}$. $\square$

Remark 1.3. Notice that $T(a,b)$, when written as $\binom{6}{a}\binom{6}{b}\binom{6}{6-a-b}$ looks a lot like Vandermonde's Theorem, with the exception that now there are 3 binomial coefficients instead of 2. However, we notice that the same argument we used there can be used here, except now we're choosing 6 stones out of 6 red, 6 blue, and 6 green stones, for a total of $\binom{18}{6}$

Finally, let's use combinatorial arguments to tackle a proof problem:

Example 1.4 (China TST 2014/2/1)

Prove that for any positive integers k and N,

$$\left(\frac{1}{N}\sum_{n=1}^{N}(\omega(n))^k\right)^{\frac{1}{k}} \leq k + \sum_{q \leq N} \frac{1}{q},$$

where $\sum\limits_{q \leq N} \frac{1}{q}$ is the summation over of prime powers $q \leq N$ (including $q=1$). Note: For integer $n > 1$, $\omega(n)$ denotes number of distinct prime factors of n, and $\omega(1) = 0$.

Solution. We start by raising both sides to the kth power, so we get

$$\frac{1}{N}\sum_{n=1}^{N}(\omega(n))^k \leq \left(k + \sum_{q\leq N}\frac{1}{q}\right)^k$$

Now, we want to find out what $(\omega(n))^k$ counts. Note that $\omega(n)$ is the number of distinct prime divisors, so $\omega(n)^k$ simply counts the number of possible k-tuples of distinct prime divisors of n.

Though this isn't super easy to evaluate either, we can sum swap. Rather than counting the number of k-tuples for n, we count number of n for each k-tuple. In particular, we fix a set of primes which the k-tuple uses, and count the number of n which can use the generated k-tuples.

So, if we have a k-tuple which uses the primes $q_1, q_2, \ldots, q_i$, then the number of n which can count it is $\left\lfloor \frac{N}{q_1 \cdots q_i} \right\rfloor$ since they need to be divisible by all the primes. Now, we need to count the number of ways to generate tuples using exactly this set of primes. This is hard to count in general, however we can upperbound it by $\binom{k-1}{i-1}\cdot k!$, since there are $\binom{k-1}{i-1}$ to choose how many of each prime we have by Sticks and Stones, and then at most $k!$ ways to permute. So, our sum is at most

$$\frac{1}{N}\sum_{n=1}^{N}(\omega(n))^k \leq \frac{1}{N}\sum_{q_1\ldots q_i\leq N}\left\lfloor\frac{N}{q_1\ldots q_i}\right\rfloor k!\binom{k-1}{i-1} \leq \sum_{q_1\ldots q_i \leq N}\frac{1}{q_1\ldots q_i}k!\binom{k-1}{i-1}$$

Now, it's time to look at the RHS. In order to show this inequality, it would suffice to show that the coefficient of $\frac{1}{q_1\cdots q_i}$ is bigger on the RHS for all $q_1 \ldots q_i$.

By Multinomial Theorem, the coefficient on the RHS is $k^{k-i}\cdot\frac{k!}{(k-i)!}$ So, we wish to show

$$k!\binom{k-1}{i-1} \leq k^{k-i}\cdot\frac{k!}{(k-i)!} \iff \frac{(k-1)!}{(i-1)!} \leq k^{k-i}$$

However, this is clear because the left hand side consists of $k-i$ factors, all of which are less than k. □

§2 Generating Functions

Given a sequence $a_0, a_1, a_2, \ldots$, the **generating function** of the sequence is

$$a_0 + a_1x + a_2x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i.$$

Generating functions are most commonly used to find the value of some term in a sequence. By writing a generating function for the sequence, we can determine the coefficient of the x^i to find the ith term in the sequence.

The generating function of an infinite sequence is also known as a **power series**. An important generating function to know is

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

Another useful generating function expansion is

$$\binom{k-1}{k-1} + \binom{k}{k-1}x + \binom{k+1}{k-1}x^2 + \cdots = \frac{1}{(1-x)^k}.$$

We can prove this result with induction on k. The last generating function that we will state (without proof) is the generating for the Fibonacci numbers:

$$x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots = \frac{x}{1-x-x^2}.$$

A hint for how to prove this is to consider the recursion $F_{n+2} = F_n + F_{n+1}$. The process for this derivation is similar to the process of deriving the generating function for the Catalan numbers in the example below.

In problems that can be solved with generating functions, the key ideas are to clearly write down the necessary generating functions, multiply them carefully and cancelling terms (if necessary), and computing the coefficients of the resulting generating functions. Many times this last step can be simplified by one of the well known generating functions above or there is a simple closed form.

§2.1 Basic Generating Functions

Example 2.1 (2007 HMMT C9)

Let $S = \{(i,j,k) \mid i+j+k = 17, i,j,k \geq 1\}$. Find

$$\sum_{(i,j,k)\in S} ijk.$$

Solution. Note that we are are given i, j, and k have a fixed sum of 17. Therefore, this motivates us to look at the coefficient of x^{17} in the generating function for this sum. To determine the generating for this sum, note the following:

$$\left(\sum_{i\geq 0} ix^i\right)\left(\sum_{j\geq 0} jx^j\right)\left(\sum_{k\geq 0} kx^k\right) = \sum_{i,j,k\geq 0} ijkx^{ijk} = \sum_{n\geq 0}\left(\sum_{i+j+k=n} ijk\right)x^n$$

Therefore, we can determine the generating function for the sum by multiplying the three generating functions on the LHS. Note that each of these generating functions on the LHS is

$$x + 2x^2 + 3x^3 + \cdots = x\left(\binom{1}{1} + \binom{2}{1}x + \binom{3}{1}x^2 + \cdots\right) = \frac{x}{(1-x)^2}$$

using one of the useful expansions we stated above. Therefore,

$$\left(\sum_{i+j+k=n} ijk\right)x^n = \left(\frac{x}{(1-x)^2}\right)^3 = \frac{x^3}{(1-x)^6} = x^3\sum_{i=0}^{\infty}\binom{5+i}{5}x^i.$$

We now want the coefficient of $x^{17-3} = x^{14}$ inside the sum which is $\binom{19}{5} = \boxed{11628}$. □

Example 2.2 (Classical)
The Catalan Numbers are defined by $C_0 = 1$ and

$$C_n = C_{n-1}C_0 + C_{n-2}C_1 + \ldots + C_0C_{n-1}$$

Find x such that $C_0 + C_1x + C_2x^2 + \ldots = \frac{7}{4}$.

Solution. Denote the generating function for the Catalan numbers by $C(x)$ and note that $C_0 + C_1x + C_2x^2 + \cdots$ is precisely the generating function for the Catalan numbers. Therefore, it suffices to determine an alternate expression for the generation function of the Catalan numbers in terms of x.

Now, note that the recursion for the Catalan numbers is quite interesting because the recursion for C_n contains all the terms C_iC_j where $i + j = n - 1$. Therefore, if we were to square the generating function for the Catalan numbers, then we can obtain the terms in the recursion by grouping terms with the same degree:

$$\begin{aligned}(C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots)^2 &= C_0^2 + (C_0C_1 + C_1C_0)x + (C_0C_2 + C_1^2 + C_2C_0)x^2 + \cdots \\ &= C_1 + C_2x + C_3x^2 + \cdots \\ &= \frac{(C_0 + C_1x + C_2x^2 + \cdots) - C_0}{x} \\ &= \frac{C(x) - 1}{x}\end{aligned}$$

Therefore,

$$C(x)^2 = \frac{C(x) - 1}{x} \implies xC(x)^2 - C(x) + 1 = 0.$$

Given this, we do not actually have to solve for $C(x)$ in terms of x for this problem. We are given that $C(x) = \frac{7}{4}$, so we can substitute this in to determine the value of x.

$$\frac{49}{16}x - \frac{7}{4} + 1 = 0 \implies x = \boxed{\frac{12}{49}}.$$

Now, since the Catalan numbers are so important, we will go further and determine an expression for the generating function of the Catalan numbers. We have already determined quadratic in $C(x)$, which we can solve with the quadratic formula. The roots of this quadratic are

$$C(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

Now, we have to determine if we want to take the positive root or the negative root. The easiest way we can check this is by plugging in $x = 0$. In this case, the generating function has the value $C_0 = 1$, so $C(x)$ must have a limit of 1 when x approaches 0. In particular, note that

$$\lim_{x \to 0} \frac{1 - \sqrt{1 - 4x}}{2x} = 1, \qquad \lim_{x \to 0} \frac{1 + \sqrt{1 - 4x}}{2x} = \text{does not exist}.$$

Therefore, we can conclude that the generating function for the Catalan numbers is $\frac{1-\sqrt{1-4x}}{2x}$. We can verify that plugging in $\frac{12}{49}$ yields $\frac{7}{4}$ in this equation.

□

§2.2 Snake Oil

In the last example for generating functions, we will demonstrate the powerful technique known as Snake Oil.

The Snake Oil method is generally used when you have a sum over one variable and another free variable that is not summed over. In the example below, this free variable is n. The first step in the Snake Oil method is to write a generating function for the sum that sums through possible values of n. Then, this generating function can be expressed as a double sum, so we can swtich the order of the summation. Then, we will evaluate both of the summations from the inside to the outside. This will yield the generating function for the sum, and now we can determine the general form of a coefficient in the generating function to find the value of the sum at a particular value of n. All of the steps will become more clear in the example below.

Example 2.3

For $n \geq 0$, compute

$$\sum_{k\geq 0} \binom{2k}{k}\binom{n}{k}\left(-\frac{1}{4}\right)^k.$$

Solution. We notice that n is our free variable here, so we can form a generating function for this sum that has the values for this sum at each value of n. We get

$$\sum_{n\geq 0}\left(\sum_{k\geq 0} \binom{2k}{k}\binom{n}{k}\left(-\frac{1}{4}\right)^k\right)x^n = \sum_{n\geq 0}\sum_{k\geq 0} \binom{2k}{k}\binom{n}{k}\left(-\frac{1}{4}\right)^k x^n$$

Now, we can switch the order of the double summation.

$$\begin{aligned}\sum_{k\geq 0}\sum_{n\geq 0} \binom{2k}{k}\binom{n}{k}\left(-\frac{1}{4}\right)^k x^n &= \sum_{k\geq 0}\binom{2k}{k}\left(-\frac{1}{4}\right)^k \sum_{n\geq 0}\binom{n}{k}x^n \\ &= \sum_{k\geq 0}\binom{2k}{k}\left(-\frac{1}{4}\right)^k \sum_{r\geq 0}\binom{k+r}{k}x^{k+r} \\ &= \sum_{k\geq 0}\binom{2k}{k}\left(-\frac{1}{4}\right)^k \frac{x^k}{(1-x)^{k+1}} \\ &= \frac{1}{1-x}\sum_{k\geq 0}\binom{2k}{k}\left(\frac{-\frac{1}{4}x}{1-x}\right)^k\end{aligned}$$

To evaluate this sum, we will use the following claim.

Claim 2.4 — The generating function $\sum_{k\geq 0}\binom{2k}{k}y^k = \frac{1}{\sqrt{1-4y}}$.

Proof. We will present an elegant proof using the generating function for the Catalan numbers that we derived above. However, note that this claim can also be proven with the Binomial Theorem.

The closed form for the kth Catalan number is $\frac{1}{k+1}\binom{2k}{k}$, which looks quite similar to the term inside the sum. We have already determined the generating function for the

Catalan numbers. Therefore,

$$\sum_{k\geq 0}\frac{1}{k+1}\binom{2k}{k}y^k=\frac{1-\sqrt{1-4y}}{2y}\implies\sum_{k\geq 0}\frac{1}{k+1}\binom{2k}{k}y^{k+1}=\frac{1-\sqrt{1-4y}}{2}$$

Now, note that if we take a derivative of the LHS, the $k+1$ cancels, and we obtain the sum in the claim. Therefore,

$$\sum_{k\geq 0}\binom{2k}{k}y^k=\frac{d}{dy}\left(\frac{1-\sqrt{1-4y}}{2}\right)=\frac{1}{\sqrt{1-4y}}.$$

Remark 2.5. There is an alternate proof with the Binomial Theorem. Can you find it?

□

Returning to the main problem, we can use the claim to obtain

$$\begin{aligned}\frac{1}{1-x}\sum_{k\geq 0}\binom{2k}{k}\left(\frac{-\frac{1}{4}x}{1-x}\right)^k&=\frac{1}{1-x}\frac{1}{\sqrt{1-4\cdot\left(\frac{-\frac{1}{4}x}{1-x}\right)}}\\&=\frac{1}{1-x}\frac{1}{\sqrt{1+\frac{x}{1-x}}}\\&=\frac{1}{1-x}\frac{1}{\sqrt{\frac{1}{1-x}}}\\&=\frac{1}{1-x}\sqrt{1-x}=\frac{1}{\sqrt{1-x}}\end{aligned}$$

Now, it suffices to determine the coefficient of x^n in the generating function for $\frac{1}{\sqrt{1-x}}$. We note that we can use the claim above where $y=\frac{x}{4}$. Therefore, we have

$$\frac{1}{\sqrt{1-x}}=\sum_{n\geq 0}\binom{2n}{n}\left(\frac{x}{4}\right)^n=\sum_{n\geq 0}\frac{\binom{2n}{n}}{4^n}x^n.$$

This means that the value of the sum is $\boxed{\frac{\binom{2n}{n}}{4^n}}$.

□

§3 Number Theoretic Sums

Some sums you encounter may involve the use of number theoretic functions, so it is always helpful to be familiar with the more common ones. We review a few here:

- $\lfloor\cdot\rfloor$: The floor function is the greatest integer less than or equal to the argument. In number theoretic sums, they typically appear as $\left\lfloor\frac{n}{k}\right\rfloor$. In this context, they count the number of multiples of k less than or equal to n.

- $\varphi(n)$: Known as Euler's Totient Function, this counts the number of positive integers $\leq n$ which are relatively prime to it. For $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, we have that
$$\varphi(n) = \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right) \cdot n$$
You can prove this easily by considering the probability of choosing a number not divisible by p_i, and noting that such probabilities are independent over all i
- $\tau(n)$ or $d(n)$: The number of divisors of n. For $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, we have
$$\tau(n) = (1 + \alpha_1)(1 + \alpha_2) \cdots (1 + \alpha_k)$$
This is because we have $1 + \alpha_i$ possible choices for the exponent of p_i for each i
- $\sigma(n)$: The sum of the divisors of n. For $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, we have
$$\sigma(n) = (1 + p_1 + \ldots p_1^{\alpha_1}) \cdots (1 + p_k + \ldots + p_k^{\alpha_k}) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdot \ldots \cdot \frac{p_k^{\alpha_k+1} - 1}{p_k - 1}$$
One can prove this just by expanding the first form and seeing that every divisor does indeed appear exactly once in the expansion

With these definitions, we can use normal sum skills to solve many sums with these functions. As there is little difference between this class of sums and what we have already looked at, we'll just give one short example:

Example 3.1 (2019 CMIMC Algebra #7)

For all positive integers n, let
$$f(n) = \sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor^2.$$
Compute $f(2019) - f(2018)$. Here $\varphi(n)$ denotes the number of positive integers less than or equal to n which are relatively prime to n.

Solution. The key for this question is to note that $\left\lfloor \frac{n}{k} \right\rfloor$ doesn't change much when n varies. In particular, it counts the number of multiples of k under n, so it should only change if n is a multiple of k, since otherwise we don't include any new multiples of k. Thus, the only terms that change between $f(2019)$ and $f(2018)$ are $k = 1, 3, 673, 2019$, making the answer
$$\varphi(1) \cdot (2019^2 - 2018^2) + \varphi(3) \cdot (673^2 - 672^2) + \varphi(673) \cdot (3^2 - 2^2) + \varphi(2019)(1^2 - 0^2)$$
$$= 4037 + 2 \cdot 1345 + 672 \cdot 5 + 1344 = \boxed{11431}$$
□

Unfortunately, most number theoretic sums are not as easy as the above. Luckily, many can be killed with one very powerful, very useful trick: Dirichlet Convolution.

Dirichlet Convolution is a convolution of two functions $f, g : \mathbb{N} \to \mathbb{R}$ defined as
$$f * g(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$
This form may seem a bit strange, however there is one main fact about convolution which makes it so useful:

Theorem 3.2

If f, g are multiplicative (that is, $f(x)f(y) = f(xy)$ if x, y are coprime, and similarly for g), then $f * g$ is also multiplicative.

Proof. Denote $c = f * g$. We wish to show that, for all pairs of coprime (x, y), we have that $c(x)c(y) = c(xy)$. Well, if we write out the LHS explicitly, we get that

$$c(x)c(y) = \left(\sum_{x|n} f(d)g\left(\frac{x}{d}\right)\right)\left(\sum_{d'|y} f(d')g\left(\frac{y}{d'}\right)\right)$$

However, since x, y are coprime, their factors do not interact, so any factor of xy, call it k, can be broken down uniquely into $k = dd'$ such that $d|x$, $d'|y$. And, this representation occurs exactly once in the expansion of the above product, since we have

$$f(d)g\left(\frac{x}{d}\right)f(d')g\left(\frac{y}{d'}\right) = f(dd')g\left(\frac{xy}{dd'}\right) = f(k)g\left(\frac{xy}{k}\right)$$

Thus, we see that a term of the form $f(k)g\left(\frac{xy}{k}\right)$ is represented exactly once in the product for all $k|xy$, and therefore we have $c(x)c(y) = c(xy)$ as desired. □

So, if we notice that our sum can be written as the convolution of two multiplicative functions, then this tells us that our sum itself is multiplicative. The reason why knowing that a function is multiplicative is so powerful is that, if we have $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ for primes p_i, then we can write $f(n) = f\left(p_1^{\alpha_1}\right) \cdots f\left(p_k^{\alpha_k}\right)$. Thus, we only need to understand our sum when n is a prime power in order to evaluate it for general n.

Let's see how noticing a convolution works in practice with the following two examples:

Example 3.3 (2015 PUMaC NT #6)

For a positive integer n, let $d(n)$ be the number of positive divisors of n. What is the smallest positive integer n such that

$$\sum_{t|n} d(t)^3$$

is divisible by 35?

Solution. First, note that $d(t)$ is multiplicative, so $d(t)^3$ must be as well. Now, the sum we have is $1 * d(t)^3$, so it must be multiplicative by Dirichlet Convolution. So, we only have to consider what will happen when n is a prime power.

If $n = p^k$, then

$$\sum_{t|n} d(t)^3 = 1^3 + 2^3 + \ldots + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

So, in order to have a sum for n divisible by 35, we must have a prime which produces a factor divisible by 5 and another one which produces one divisible by 7 (note that having one which produces a factor of 35 is unviable since this is too large.) To take care of 5 we can choose $k = 3$ and for 7 we choose $k = 5$. Thus, the minimum n is $n = 2^5 \cdot 3^3 = \boxed{864}$. □

Example 3.4 (Bulgaria 1989)

Let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity. Evaluate

$$\sum_{n=1}^{1989}(-1)^{\Omega(n)}\left\lfloor\frac{1989}{n}\right\rfloor$$

Solution. Note that $(-1)^{\Omega(n)}$ is indeed a multiplicative function (in fact, it is completely multiplicative.) However, we don't have the correct form of convolution to assume that the current sum is multiplicative. However, recall that $\left\lfloor\frac{1989}{n}\right\rfloor$ is the number of multiples of n under 1989, so using that, we get:

$$\sum_{n=1}^{1989}(-1)^{\Omega(n)}\left\lfloor\frac{1989}{n}\right\rfloor=\sum_{n=1}^{1989}(-1)^{\Omega(n)}\sum_{n|k,k\leq 1989}1=\sum_{k=1}^{1989}\sum_{n|k}(-1)^{\Omega(n)}$$

Aha! The inner sum is now a convolution $1*f$, where $f(n)=(-1)^{\Omega(n)}$. When $n=p^k$, $1*f$ is 1 if k is even, and 0 otherwise. So, the inner sum will evaluate to 1 if and only if all primes have even exponent, and 0 otherwise. However, the former is equivalent to n being a square! So, our original sum just counts the number of squares ≤ 1989.

Thus, our answer is $\lfloor\sqrt{1989}\rfloor=\boxed{44}$. □

§4 Problems

Problem 4.1 (2010 MPFG #16)**.** Let P be the quadratic function such that $P(0) = 7$, $P(1) = 10$, and $P(2) = 25$. If a, b, and c are integers such that every positive number x less than 1 satisfies

$$\sum_{n=0}^{\infty} P(n)x^n = \frac{ax^2 + bx + c}{(1-x)^3}$$

compute the ordered triple (a, b, c).

(Note: This was a class example last week. See if you can solve it with generating functions for a quicker solution.)

Problem 4.2. Let $m \le n$ be positive integers. Compute

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m}.$$

Problem 4.3 (AoPS)**.** Three of my friends and I are going to split the bill for dinner. Adithya and I will each contribute an odd number of dollars, while William contributes a number of dollars that is a multiple of 3. Brandon will either contribute nothing, or steal one or two dollars. In how many ways can we pay a $30 bill?

Problem 4.4. Show that

$$\sum_{d|n} \varphi(d) = n$$

Problem 4.5 (Strehl)**.** Show that

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{k=0}^{n} \binom{n}{k}^k \binom{2k}{n}$$

Problem 4.6. a) Prove that the generating function for the Fibonacci numbers is $\frac{x}{1-x-x^2}$

b) Find a closed form for the Fibonacci numbers with this generating function

Problem 4.7. For $m, n \ge 1$ compute

$$\sum_{k \ge 0} \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}.$$

Problem 4.8 (2011 PUMaC NT #5)**.** Let $d(n)$ denote the number of divisors of n (including itself). You are given that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Find $p(6)$, where $p(x)$ is the unique polynomial with rational coefficients satisfying

$$p(\pi) = \sum_{n=1}^{\infty} \frac{d(n)}{n^2}$$

Problem 4.9. For $n \geq 1$, evaluate the sum

$$\sum_{k \geq 0} \binom{n+k}{2k} 2^{n-k}.$$

Problem 4.10. Evaluate the following sum when $y = \pm 2$:

$$\sum_{k \geq 0} \binom{n}{k} \binom{n-k}{\lfloor \frac{m-k}{2} \rfloor} y^k$$

Problem 4.11 (2019 HMMT Algebra and NT #8)**.** There is a unique function $f : \mathbb{N} \to \mathbb{R}$ such that $f(1) > 0$ and such that

$$\sum_{d|n} f(d) f\left(\frac{n}{d}\right) = 1$$

for all $n \geq 1$. What is $f\left(2018^{2019}\right)$

Problem 4.12 (2008 HMMT Algebra #10)**.** Evaluate the infinite sum

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}$$

(Hint: it may be helpful to know that the nth Catalan Number has formula $C_n = \frac{1}{n+1}\binom{2n}{n}$)

Problem 4.13 (Moriati)**.** For given n and p evaluate

$$\sum_{k \geq 0} \binom{2n+1}{2p+2k+1} \binom{p+k}{k}.$$

Problem 4.14 (USA TST 2010/8)**.** Let m, n be positive integers with $m \geq n$, and let S be the set of all n-term sequences of positive integers $(a_1, a_2, \dots a_n)$ such that $a_1 + a_2 + \cdots + a_n = m$. Show that

$$\sum_S 1^{a_1} 2^{a_2} \cdots n^{a_n} = \binom{n}{n} n^m - \binom{n}{n-1} (n-1)^m + \cdots + (-1)^{n-2} \binom{n}{2} 2^m + (-1)^{n-1} \binom{n}{1}.$$

Problem 4.15 (2015 OMO #30)**.** Ryan is learning number theory. He reads about the Möbius function $\mu : \mathbb{N} \to \mathbb{Z}$, defined by $\mu(1) = 1$ and

$$\mu(n) = -\sum_{\substack{d|n \\ d \neq n}} \mu(d)$$

for $n > 1$ (here $\mathbb{N}$ is the set of positive integers). However, Ryan doesn't like negative numbers, so he invents his own function: the dubious function $\delta : \mathbb{N} \to \mathbb{N}$, defined by the relations $\delta(1) = 1$ and

$$\delta(n) = \sum_{\substack{d|n \\ d \neq n}} \delta(d)$$

for $n > 1$. Help Ryan determine the value of $1000p + q$, where p, q are relatively prime positive integers satisfying

$$\frac{p}{q} = \sum_{k=0}^{\infty} \frac{\delta(15^k)}{15^k}.$$