Summations A

Adithya B., Brian L., William W., Daniel X.

5/27

§1 Basic Sums

We'll first review a couple of the most basic types of sums: arithmetic and geometric series. In particular, you should be familiar with the following identities:

- Sum of first n terms of arithmetic sequence $a, a+d, \ldots = \frac{n(2a+(n-1)d)}{2}$
- Sum of first n terms of geometric sequence $a, ar, \ldots = a\frac{r^n-1}{r-1}$
- Sum of infinite geometric sequence $a, ar, \ldots$ with $|r| < 1 = \frac{a}{1-r}$

For a quick proof of the formula for the arithmetic sequence, note that the sum of "opposite pairs" (i.e. 1st and last, 2nd and 2nd to last) is always $2a + (n-1)d$, so we multiply by $\frac{n}{2}$ to recover the identity. For the formulae concerning geometric sequences, multiply each term by r and subtract the resultant sum from the initial one.

Since there is not too much complexity involving these sums, we'll only go through one brief example:

Example 1.1 (2010 MPFG #16)

Let P be the quadratic function such that $P(0) = 7$, $P(1) = 10$, and $P(2) = 25$. If a, b, and c are integers such that every positive number x less than 1 satisfies

$$\sum_{n=0}^{\infty} P(n)x^n = \frac{ax^2+bx+c}{(1-x)^3},$$

compute the ordered triple (a, b, c).

Solution. First, we find $P(x)$. $P(0) = 7$ tells us that it is of the form $mx^2 + nx + 7$. We have $m + n = 3, 4m + 2n = 18$, so $m = 6, n = -3$, and $P(x) = 6x^2 - 3x + 7$. So,

$$\sum_{n=0}^{\infty} P(n)x^n = 6\sum_{n=0}^{\infty} n^2x^n - 3\sum_{n=0}^{\infty} nx^n + 7\sum_{n=0}^{\infty} x^n$$

We will evaluate each sum independently. The easiest one is the last one, since we recognize that it is just $\frac{1}{1-x}$ by our formula for infinite geometric sequence.

For the second, if $S = \sum_{n=0}^{\infty} nx^n$, note that

$$xS = \sum_{n=1}^{\infty}(n-1)x^n \implies (1-x)S = \sum_{n=1}^{\infty} nx^n - (n-1)x^n = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$$

So, $S = \frac{x}{(1-x)^2}$.

Finally, for the last sum, if we denote $T = \sum_{n=0}^{\infty} n^2x^n$, note that

$$xT = \sum_{n=1}^{\infty}(n-1)^2x^n \implies (1-x)T = \sum_{n=1}^{\infty} x^n(n^2-(n-1)^2) = \sum_{n=1}^{\infty}(2n-1)x^n$$

$$= 2\sum_{n=1}^{\infty} nx^n - \sum_{n=1}^{\infty} x^n = 2S - \frac{x}{1-x} = \frac{x^2+x}{(1-x)^2}$$

So, $T = \frac{x^2+x}{(1-x)^3}$ and our final answer is

$$6T - 3S + \frac{7}{1-x} = \frac{6(x^2+x) - 3(x-x^2) + 7(x^2-2x+1)}{(1-x)^3} = \frac{16x^2-11x+7}{(1-x)^3}$$

So, our final answer is $\boxed{(16, -11, 7)}$. □

Remark 1.2. The above solution demonstrates the power of multiplying by the common ratio and subtracting for geometric-like summations. For readers who know calculus, note that the formulae for S and T can be derived easily by differentiation. For example,

$$S = \sum_{n=0}^{\infty} nx^n = x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n = x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{x}{(1-x)^2}$$

See if you can recover the formula for T by differentiating the sum S using a similar method as above.

§2 Telescoping

Often times, complicated sums can be simplified by noting an identity which will allow for a lot of canceling. In particular, if we can write a sum in the form

$$\sum_{i=1}^{n} f(i+1) - f(i)$$

for some function f, then note that $f(i)$ and $-f(i)$ appear for $2 \le i \le n$, so $f(i)$ will cancel at all of these values. Hence, we can ignore everything in the middle, meaning

$$\sum_{i=1}^{n} f(i+1) - f(i) = f(n+1) - f(1)$$

As the sum collapses inward on itself like a telescope, this method is called telescoping. Of course, if the summand were written as $f(i+1) - f(i)$, it is very easy to telescope, however often times we need to first manipulate it into this form. A couple common techniques include:

- Partial fraction decomposition: If the summand has a factorable denominator, it can often times be telescoped. For example, the fraction $\frac{1}{n(n+1)}$ can be broken down as $\frac{1}{n} - \frac{1}{n+1}$, from which the telescope is evident. A less obvious example is $\frac{n}{n^4+n^2+1}$, where one needs to recognize the denominator as $(n^2-n+1)(n^2+n+1)$.

- Trigonometric identities: Differences in trig can be easily masked by sum to product. For example, the product $\sin(1)\cos(n+1)$ actually is equal to $\frac{\sin(n+2)-\sin(n)}{2}$. This idea is relevant for evaluating sums of the form $\sin(a)+\sin(a+d)+\ldots+\sin(a+kd)$. See how it telescopes when we multiply by $\sin\left(\frac{d}{2}\right)$.
- Ad hoc: Most telescoping doesn't have a particular method. Whenever you see a sum, though, always think for a second to see how you can write it as a difference. Not all sums can be telescoped, but noticing that one can will vastly decrease the amount of time you take on it, since telescoping is generally a one-step solve.

Example 2.1 (2002 AIME I #4)

Consider the sequence defined by $a_k = \frac{1}{k^2+k}$ for $k \geq 1$. Given that $a_m + a_{m+1} + \cdots + a_{n-1} = 1/29$, for positive integers m and n with $m < n$, find $m+n$.

Solution. We can telescope each term of the sequence. We have $a_k = \frac{1}{k} - \frac{1}{k+1}$. Therefore,

$$a_m + a_{m+1} + \cdots + a_{n-1} = \frac{1}{m} - \frac{1}{m+1} + \frac{1}{m+1} - \frac{1}{m+2} + \cdots + \frac{1}{n-1} - \frac{1}{n} = \frac{1}{m} - \frac{1}{n}.$$

Therefore, $\frac{1}{m} - \frac{1}{n} = \frac{1}{29}$. We can clear the denominators by multiplying by $29mn$. Therefore,

$$mn = 29n - 29m \implies mn + 29m - 29n = 0$$

We see that if we subtract $29^2 = 841$ from both sides, then we can factor the equation.

$$mn + 29m - 29n - 841 = -841 \implies (m-29)(n+29) = -841$$

Since $m < n$ and $m, n > 0$, we must have $m - 29 = -1$ and $n + 29 = 841$. Therefore, we get $m = 28$ and $n = 812$. This means that $m + n = \boxed{840}$. □

Example 2.2 (MOP)

Compute the sum

$$\sum_{k=0}^{n} \frac{(4k+1)k!}{(2k+1)!}$$

Solution. A good way to start telescoping is to try to note the relationship between consecutive terms. Note that

$$\frac{k!}{(2k+1)!} = \frac{(k+1)!}{(2k+3)!} \cdot \frac{(2k+3)(2k+2)}{(2k+1)} = (4k+6) \cdot \frac{(k+1)!}{(2k+3)!}$$

This means that we can write $\frac{(4k+1)k!}{(2k+1)!}$ as

$$\frac{(4k+2)k!}{(2k+1)!} - \frac{k!}{(2k+1)!} = \frac{(4k+2)k!}{(2k+1)!} - \frac{(4k+6)(k+1)!}{(2k+3)!}$$

Now, note that if we define $f(k) = \frac{(4k+2)k!}{(2k+1)!} = \frac{2 \cdot k!}{(2k)!} = \frac{(k-1)!}{(2k-1)!}$, this expression is just $f(k) - f(k+1)$. So, we have found a telescope!

Our final answer, then, is

$$\boxed{f(0) - f(n+1) = 2 - \frac{n!}{(2n+1)!}}$$

□

§3 Double Sums

Sometimes, we are asked to compute *double* sums, in which the terms of one summation are sums themselves. An example is

$$\sum_{a=0}^{\infty}\left(\sum_{b=0}^{\infty}\frac{1}{2^{2a+b}}\right).$$

To evaluate this, we would need to evaluate the inner sum in terms of a:

$$\frac{1}{2^{2a}}+\frac{1}{2^{2a+1}}+\frac{1}{2^{2a+2}}+\cdots=\frac{1}{2^{2a-1}}$$

and then feed this back into the outer sum:

$$\frac{1}{2^{-1}}+\frac{1}{2^{1}}+\frac{1}{2^{3}}+\frac{1}{2^{5}}+\cdots=\frac{8}{3}.$$

One very powerful technique in evaluating double sums is *swapping the order of summation*, i.e.

$$\sum_{a=0}^{\infty}\left(\sum_{b=0}^{\infty}\frac{1}{2^{2a+b}}\right)=\sum_{b=0}^{\infty}\left(\sum_{a=0}^{\infty}\frac{1}{2^{2a+b}}\right).$$

However, double sums aren't always immediately presented in a format that makes the above formula directly applicable. When approaching something that could potentially be a double sum, it's more important to *look at the sum from multiple perspectives.* We should consider various orders of summing the terms other than the order presented to us (which is essentially what "swapping the order of summation" is).

Example 3.1 (2017 HMMT Algebra and Number Theory #5)

Kelvin the Frog was bored in math class one day, so he wrote all ordered triples (a,b,c) of positive integers such that $abc=2310$ on a sheet of paper. Find the sum of all integers he wrote down. In other words, compute

$$\sum_{\substack{abc=2310\\ a,b,c\in\mathbb{N}}}(a+b+c),$$

where $\mathbb{N}$ denotes the positive integers.

Solution. In this problem, we're asked to find the sum of all $a+b+c$. Instead of computing the values of $a+b+c$ and summing them, it's more natural to consider how many times Kelvin wrote down a given number. After all, the given sum is just the total of all the numbers Kelvin wrote down.

So let's find how many times Kelvin wrote down a given n. Clearly all the numbers that Kelvin writes are divisors of 2310, so we assume that n is a divisor of 2310. For how many triples (a,b,c) does $a=n$?. In this case, we must have $bc=\frac{2310}{n}$. Thus the number of choices for b and c is equal to the number of positive divisors of $\frac{2310}{n}$, which we denote by $\tau\left(\frac{2310}{n}\right)$. Hence $n=a$ a total of $\tau\left(\frac{2310}{n}\right)$ times. Then the total number of times Kelvin writes down n is

$$3\tau\left(\frac{2310}{n}\right),$$

as the number of times Kelvin wrote down n as the first number of the triple is equal to the number of times he wrote n as the second number as well as the number of times he wrote n as the third number. Thus the sum of all of the numbers that Kelvin writes is

$$\sum_{n|2310} 3n\tau\left(\frac{2310}{n}\right).$$

Using $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, a computation yields the answer of $\boxed{49140}$. □

Example 3.2 (2017 PUMaC Algebra A #7)
Compute

$$\sum_{k=0}^{\infty} \frac{2^k}{5^{2^k}+1}.$$

Solution. At first glance, we're given a single sum that doesn't simplify nicely algebraically: there's nothing immediately obvious we can do with the sequence

$$\frac{1}{1+5^1} + \frac{2}{1+5^2} + \frac{4}{1+5^4} + \frac{8}{1+5^8} + \frac{16}{1+5^{16}} + \cdots.$$

However, there's something that we can do with fractions of the form $1 + \text{something}$: expand it as a geometric series:

$$\frac{1}{1+5^1} = \frac{1}{5} - \frac{1}{5^2} + \frac{1}{5^3} - \cdots.$$

If we do this to all of the terms of the series, we get a large double sum:

$$\begin{aligned}
\frac{1}{1+5^1} &= \frac{1}{5^1} - \frac{1}{5^2} + \frac{1}{5^3} - \frac{1}{5^4} + \frac{1}{5^5} - \frac{1}{5^6} + \frac{1}{5^7} - \frac{1}{5^8} + \cdots \\
\frac{2}{1+5^2} &= \phantom{\frac{1}{5^1} - {}} \frac{2}{5^2} \phantom{{}+\frac{1}{5^3}} - \frac{2}{5^4} \phantom{{}+\frac{1}{5^5}} + \frac{2}{5^6} \phantom{{}+\frac{1}{5^7}} - \frac{2}{5^8} + \cdots \\
\frac{4}{1+5^4} &= \phantom{\frac{1}{5^1} - \frac{1}{5^2} + \frac{1}{5^3} - {}} \frac{4}{5^4} \phantom{{}+\frac{1}{5^5}-\frac{1}{5^6}+\frac{1}{5^7}} - \frac{4}{5^8} + \cdots \\
\frac{8}{1+5^8} &= \phantom{\frac{1}{5^1} - \frac{1}{5^2} + \frac{1}{5^3} - \frac{1}{5^4} + \frac{1}{5^5} - \frac{1}{5^6} + \frac{1}{5^7} - {}} \frac{8}{5^8} - \cdots
\end{aligned}$$

Now if we add all of these infinite series, something remarkable happens: the sum simplifies to

$$\frac{1}{5^1} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots = \boxed{\frac{1}{4}}.$$

More specifically, if we look down the columns, then the coefficient of $\frac{1}{5^n}$ of the sum is equal to 1 for any positive integer n. If the largest power of 2 dividing n is 2^k, then the coefficient of $\frac{1}{5^n}$ is

$$-1 - 2 - 4 - \cdots - 2^{n-1} + 2^n = 1.$$

□

Another way we can handle double sums is by *factoring* the double sum into the product of two sums, i.e.

$$\sum_{a\in A, b\in B} ab = \left(\sum_{a\in A} a\right)\left(\sum_{b\in B} b\right).$$

Example 3.3 (2017 HMMT Algebra and Number Theory #2)

Find the value of

$$\sum_{1\le a<b<c} \frac{1}{2^a 3^b 5^c}$$

(i.e. the sum of $\frac{1}{2^a 3^b 5^c}$ over all triples of positive integers (a, b, c) satisfying $a < b < c$).

Solution. The constraint that $a < b < c$ seems hard to handle directly, so we'll let $b = a + x$, and $c = a + x + y$. Now, we have the much simpler constraint that $a, x, y \ge 1$. Therefore, the sum becomes

$$\begin{aligned}\sum_{1\le a<b<c} \frac{1}{2^a 3^b 5^c} &= \sum_{a=1}^{\infty}\sum_{x=1}^{\infty}\sum_{y=1}^{\infty} \frac{1}{2^a 3^{a+x} 5^{a+x+y}} \\ &= \sum_{a=1}^{\infty}\sum_{x=1}^{\infty}\sum_{y=1}^{\infty} \frac{1}{2^a 3^a 5^a}\frac{1}{3^x 5^x}\frac{1}{5^y} \\ &= \sum_{a=1}^{\infty}\sum_{x=1}^{\infty}\sum_{y=1}^{\infty} \frac{1}{30^a}\frac{1}{15^x}\frac{1}{5^y}\end{aligned}$$

Now, we can factor the triple sum into the product of three single sums:

$$\left(\sum_{a=1}^{\infty}\frac{1}{30^a}\right)\left(\sum_{x=1}^{\infty}\frac{1}{15^x}\right)\left(\sum_{y=1}^{\infty}\frac{1}{5^y}\right) = \frac{\frac{1}{30}}{1-\frac{1}{30}}\frac{\frac{1}{15}}{1-\frac{1}{15}}\frac{\frac{1}{5}}{1-\frac{1}{5}} = \boxed{\frac{1}{1624}}.$$

□

Example 3.4 (2013 HMMT Algebra #7)

Compute

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1+a_2+\cdots+a_7}{3^{a_1+a_2+\cdots+a_7}}.$$

Solution. We consider the simpler sum

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1}{3^{a_1+a_2+\cdots+a_7}}.$$

Notice that it should be equal to one-seventh the original sum, by symmetry, and the numerator is now much easier to deal with. In fact, we note that it can actually be factored as

$$\left(\sum_{a_1=0}^{\infty}\frac{a_1}{3^{a_1}}\right)\left(\sum_{a_2=0}^{\infty}\frac{1}{3^{a_2}}\right)\left(\sum_{a_3=0}^{\infty}\frac{1}{3^{a_3}}\right)\cdots\left(\sum_{a_7=0}^{\infty}\frac{1}{3^{a_7}}\right).$$

Now, note that each sum except the first one is just a geometric series with value $\frac{1}{1-\frac{1}{3}} = \frac{3}{2}$, while the first one is $\frac{\frac{1}{3}}{(1-\frac{1}{3})^2} = \frac{3}{4}$, recalling a result found in Example 1.1. Thus, we find that our desired sum is $7\left(\frac{3}{4}\right)\left(\frac{3}{2}\right)^6 = \boxed{\frac{15309}{256}}$.

Remark 3.5. Alternatively, one can write the 7-sum as the single sum

$$\sum_{n=0}^{\infty}\binom{n+6}{6}\frac{n}{3^n}=7\sum_{n=0}^{\infty}\binom{n+6}{7}\frac{1}{3^n}=\frac{7}{3}\sum_{n=-1}^{\infty}\binom{n+7}{7}\frac{1}{3^n}$$

where $n = a_1 + a_2 + \cdots + a_7$. If you're familiar with generating functions, we can use

$$\frac{1}{(1-x)^8}=\sum_{n=0}^{\infty}\binom{n+7}{7}x^n \implies \sum_{n=0}^{\infty}\binom{n+7}{7}\frac{1}{3^n}=\frac{1}{(1-\frac{1}{3})^8}=\frac{6561}{256}$$

for the answer of

$$\frac{7}{3}\left(\binom{-1+7}{7}\frac{1}{3^{-1}}+\frac{6561}{256}\right)=\frac{15309}{256}.$$

□

§4 Ad Hoc

Lastly, we discuss ad hoc sums problems that don't fall into any of the above categories. The main theme of these types of problems is careful computation: the ideas are not hard, but it is important to describe the terms of the sum carefully, as displayed in the following examples. There are several other problems of this flavor in the exercises.

Example 4.1 (APMO 2000/1)

Compute the sum: $\sum_{i=0}^{101} \frac{x_i^3}{1-3x_i+3x_i^2}$ for $x_i = \frac{i}{101}$.

Notice that the denominator looks sort of like $(1-x_i)^3$. This motivates us to try to write the denominator in this form. The denominator is $(1-x_i)^3+x_i^3 = x_{101-i}^3+x_i^3$. Therefore, the sum can be rewritten as $\sum_{i=0}^{101}\frac{x_i^3}{x_{101-i}^3+x_i^3}$. But now, notice that $\frac{x_i^3}{x_{101-i}^3+x_i^3}+\frac{x_{101-i}^3}{x_{101-i}^3+x_i^3}=1$. Therefore, we can pair up the terms so that they add to 1. The sum becomes

$$\sum_{i=0}^{101}\frac{x_i^3}{x_{101-i}^3+x_i^3}=\sum_{i=0}^{50}\left(\frac{x_i^3}{x_{101-i}^3+x_i^3}+\frac{x_{101-i}^3}{x_{101-i}^3+x_i^3}\right)=\boxed{51}.$$

Example 4.2 (2007 AIME I #11)

For each positive integer p, let $b(p)$ denote the unique positive integer k such that $|k-\sqrt{p}|<\frac{1}{2}$. For example, $b(6)=2$ and $b(23)=5$. If $S=\sum_{p=1}^{2007} b(p)$, find the remainder when S is divided by 1000.

We need to determine $b(p)$ in order to compute the sum. The given inequality implies that if $k = b(p)$ then

$$|k-\sqrt{p}|<\frac{1}{2} \implies k-\frac{1}{2}<\sqrt{p}<k+\frac{1}{2}.$$

If we square both sides then we have the inequality

$$k^2-k+\frac{1}{4}<p<k^2+k+\frac{1}{4}.$$

Since $k = b(p) \geq 1$ for all positive integers p, we have that $k - \frac{1}{2} > 0$; thus if p satisfies the above inequality then we can reverse our steps so that $k - \frac{1}{2} < \sqrt{p} < k + \frac{1}{2}$. Hence the positive integers p for which $b(p) = k$ are precisely those positive integers p satisfying the above inequality, which are

$$k^2 - k + 1, k^2 - k + 2, \cdots, k^2 + k$$

for a total of $2k$ integers. Thus

$$b(1) = b(2) = 1, b(3) = b(4) = b(5) = b(6) = 2, \ldots.$$

Note that $b(1893) = \cdots b(1980) = 44$, so for each $k \leq 44$ there are $2k$ positive integers $1 \leq p \geq 2007$ with $b(p) = k$. Since $b(1981) = \cdots b(2007) = 45$, the sum is

$$\sum_{k=1}^{44} k \cdot 2k + \sum_{p=1981}^{2007} 45 = 2 \cdot \frac{44 \cdot 45 \cdot 89}{6} + 27 \cdot 45 = 59955$$

and the answer is $\boxed{955}$.

§5 Problems

Problem 5.1 (1984 AIME # 1)**.** Find the value of $a_2+a_4+a_6+\cdots+a_{98}$ if a_1, a_2, a_3, ... is an arithmetic progression with common difference 1, and $a_1+a_2+a_3+\cdots+a_{98}=137$.

Problem 5.2 (2016 AIME I # 1)**.** For $-1<r<1$, let $S(r)$ denote the sum of the geometric series

$$12+12r+12r^2+12r^3+\dots.$$

Let a between -1 and 1 satisfy $S(a)S(-a)=2016$. Find $S(a)+S(-a)$.

Problem 5.3 (2007 AIME I # 7)**.** Let

$$N=\sum_{k=1}^{1000} k(\lceil \log_{\sqrt{2}} k\rceil - \lfloor \log_{\sqrt{2}} k\rfloor).$$

Find the remainder when N is divided by 1000.

Problem 5.4 (2002 AIME II/6)**.** Find the integer that is closest to $1000\sum\limits_{n=3}^{10000}\frac{1}{n^2-4}$.

Problem 5.5 (2014 AIME II # 7)**.** Let $f(x)=(x^2+3x+2)^{\cos(\pi x)}$. Find the sum of all positive integers n for which

$$\left|\sum_{k=1}^{n}\log_{10} f(k)\right|=1.$$

Problem 5.6 (2018 HMMT Team # 2)**.** Is the number

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)\cdots\left(1+\frac{1}{2018}\right)$$

greater than, less than, or equal to 50?

Problem 5.7 (2006 AIME I #13)**.** For each even positive integer x, let $g(x)$ denote the greatest power of 2 that divides x. For example, $g(20)=4$ and $g(16)=16$. For each positive integer n, let $S_n=\sum_{k=1}^{2^{n-1}} g(2k)$. Find the greatest integer n less than 1000 such that S_n is a perfect square.

Problem 5.8 (2020 HMMT Algebra and Number Theory #4)**.** For positive integers n and k, let $\mho(n,k)$ be the number of distinct prime divisors of n that are at least k. For example, $\mho(90,3)=2$, since the only prime factors of 90 that are at least 3 are 3 and 5. Find the closest integer to

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\mho(n,k)}{3^{n+k-7}}.$$

Problem 5.9 (1995 AIME # 13)**.** Let $f(n)$ be the integer closest to $\sqrt[4]{n}$. Find

$$\sum_{k=1}^{1995}\frac{1}{f(k)}.$$

Problem 5.10 (AoPS)**.** Evaluate for $n\in Z_+$ sum

$$\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{z=1}^{n}\min(x,y,z)$$

Problem 5.11 (PUMaC 2017 PUMaC Algebra A #5)**.** Let $f_0(x) = x$, and for each $n \geq 0$, let $f_n(x) = f_n(x^2(3-2x))$. Find the smallest real number that is at least as large as

$$\sum_{n=0}^{2017} f_n(a) + \sum_{n=0}^{2017} f_n(1-a)$$

for all $a \in [0,1]$.

Problem 5.12 (Putnam 2015 B4)**.** Let T be the set of all triples (a,b,c) of positive integers for which there exist triangles with side lengths a,b,c. Express

$$\sum_{(a,b,c)\in T} \frac{2^a}{3^b 5^c}$$

as a rational number in lowest terms.

Problem 5.13 (2019 HMMT Algebra and Number Theory #7)**.** Find the value of

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty} \frac{ab(3a+c)}{4^{a+b+c}(a+b)(b+c)(c+a)}.$$

Problem 5.14 (2016 PUMaC Algebra #8)**.** Define the function $f : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R}$ to be

$$f(x) = \sum_{a,b=0}^{\infty} \frac{x^{2^a 3^b}}{1 - x^{2^{a+1}3^{b+1}}}$$

Suppose that $f(y) - f\left(\frac{1}{y}\right) = 2016$. Then, y can be written in simplest form as $\frac{p}{q}$. Compute $p+q$.

Problem 5.15 (2010 ISL N1)**.** Find the least positive integer n for which there exists a set $\{s_1, s_2, \ldots, s_n\}$ consisting of n distinct positive integers such that

$$\left(1 - \frac{1}{s_1}\right)\left(1 - \frac{1}{s_2}\right)\cdots\left(1 - \frac{1}{s_n}\right) = \frac{51}{2010}.$$

Problem 5.16 (USA TST 2000/4)**.** Let n be a positive integer. Prove that

$$\binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1} = \frac{n+1}{2^{n+1}}\left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1}\right).$$