Sequences

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§1 Basic Sequences

We'll start with the most basic sequences: arithmetic and geometric sequences.

Definition 1.1. An *arithmetic sequence* is a sequence in which the difference between any two consecutive terms is constant. We call this constant the *common difference* of the arithmetic sequence.

Thus the terms of a k-term arithmetic sequence can be written as

$$a,\ a+d,\ a+2d, \dots,\ a+(k-1)d$$

where a is the first term and d is the common difference.

Definition 1.2. A *geometric sequence* is a sequence in which the ratio between any two consecutive terms is (a nonzero) constant. This constant is called the *common ratio* of the sequence.

We may write the terms of a k-term geometric sequence as

$$a,\ ar,\ ar^2, \dots,\ ar^{k-1}$$

where a is the first term and r is the common ratio.

When faced with a problem dealing with an arithmetic or geometric sequence, a reliable strategy is to write write all of the terms in terms of the first term and common ratio/difference. This is because the entire sequence can be described in terms of three numbers: the first term, number of terms, and common ratio/difference. All of the information we know about the sequence can be encapsulated in a fairly small number of variables, allowing us to easily solve for the variables and thus determine the entire sequence.

Example 1.3 (2003 AIME I #8)

In an increasing sequence of four positive integers, the first three terms form an arithmetic progression, the last three terms form a geometric progression, and the first and fourth terms differ by 30. Find the sum of the four terms.

Solution. It is often a good idea to assign variables so that the expressions we obtain are as clean as possible. Since the first three terms form an arithemetic progression, we could set them to be $a, a+d, a+2d$. Since the last three terms form a geometric sequence, their common ratio is $\frac{a+2d}{a+d}$ so the fourth term is $\frac{(a+2d)^2}{a+d}$. But we can clean this up a little, by instead setting the first three terms to be $a-d, a, a+d$. Then the fourth term is $\frac{(a+d)^2}{a}$ instead. Now we have the equation

$$\frac{(a+d)^2}{a} - (a-d) = 30 \implies (a+d)^2 - a(a-d) = 30a \implies d^2 + 3ad = 30a.$$

Let's solve for a in terms of d; we try this because the above equation is a linear equation in a. We get

$$a = \frac{d^2}{30-3d}.$$

Since we know that $a, d > 0$ (the terms of the sequence are positive, and the sequence is increasing) this gives us that $d < 10$. Moreover, the denominator is divisible by 3, and as a is an integer the number must be divisibly by 3 as well. This narrows down d to $3, 6, 9$. If $d = 3$ then $a = \frac{3}{7}$, which is not an integer. IF $d = 6$ then $a = 3$, but then the first term is $a - d = -3$ which is negative. Thus we must have $d = 9$, which gives us $a = 27$. So our sequence is

$$18, 27, 36, 48$$

which we see satisfies the conditions of the problem. So the answer is $18 + 27 + 36 + 48 = \boxed{129}$.

□

Sometimes, we aren't presented with a completely arithmetic or geometric sequence. Nevertheless, we can still use the special properties of such sequences.

Example 1.4 (2004 AIME II #9)

A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $n \geq 1$, the terms a_{2n-1}, a_{2n}, a_{2n+1} are in geometric progression, and the terms a_{2n}, a_{2n+1}, and a_{2n+2} are in arithmetic progression. Let a_n be the greatest term in this sequence that is less than 1000. Find $n + a_n$.

Solution. Since the first three terms are in geometric progression, we can write them as $1, r, r^2$ for their common ratio r (which must be an integer). Since a_2, a_3, a_4 are in arithmetic progression we can write the fourth term as the third term plus the common difference, or

$$a_4 = a_3 + (a_3 - a_2) = 2a_3 - a_2 = r^2 + (r^2 - r) = 2r^2 - r.$$

Now a_3, a_4, a_5 are in geometric progression, so

$$\frac{a_4}{a_3} = \frac{a_5}{a_4} \implies a_5 = \frac{a_4^2}{a_3} = \frac{(2r^2-r)^2}{r^2} = (2r-1)^2.$$

Now a_4, a_5, a_6 form an arithmetic sequence, so

$$a_6 = 2a_5 - a_4 = 2(2r-1)^2 - r(2r-1) = (2r-1)(3r-2).$$

We can repeat this logic: since a_5, a_6, a_7 form a geometric sequence we have

$$a_7 = \frac{a_6^2}{a_5} = \frac{(2r-1)^2(3r-2)^2}{(2r-1)^2} = (3r-2)^2.$$

Continuing,

$$\begin{aligned} a_8 &= 2a_7 - a_6 = 2(3r-2)^2 - (2r-1)(3r-2) = (4r-3)(3r-2), \\ a_9 &= \frac{a_8^2}{a_7} = \frac{(4r-3)^2(3r-2)^2}{(3r-2)^2} = (4r-3)^2, \\ a_{10} &= 2a_9 - a_8 = 2(4r-3)^2 - (4r-3)(3r-2) = (4r-3)(5r-4). \end{aligned}$$

Now that we have expressions for a_9, a_10, we can write

$$646 = a_9 + a_{10} = (4r-3)^2 + (4r-3)(5r-4) = (4r-3)(9r-7).$$

Since this is a quadratic we can solve for r. Alternatively, we can factor $646 = 2 \cdot 17 \cdot 19$ and try to use this factorization to find r (since r must be an integer). Either way we can find that $r = 5$.

Now we have $a_9 = (4r-3)^2 = 289,\ a_{10} = (4r-3)(5r-4) = 357$. We can continue computing the a_i until we get the largest term less than 1000:

$$\begin{aligned} a_{11} &= \frac{a_{10}^2}{a_9} = 441, \\ a_{12} &= 2a_{11} - a_{10} = 525, \\ a_{13} &= \frac{a_{12}^2}{a_{11}} = 625, \\ a_{14} &= 2a_{13} - a_{12} = 725, \\ a_{15} &= \frac{a_{14}^2}{a_{13}} = 841, \\ a_{16} &= 2a_{15} - a_{14} = 957, \\ a_{17} &= \frac{a_{16}^2}{a_{15}} = 1089. \end{aligned}$$

So $a_{16} = 957$ is the largest term less than 1000, and the answer is $16 + 957 = \boxed{973}$. □

§2 Recursions

Broadly speaking, recursive sequences are those for which the nth term is determined based on the terms before it. As this is a very broad class of sequences, there aren't any panaceas for dealing with them. However, some ideas may help reduce work by a lot.

Sometimes, a recursion will be *periodic*, so that the sequence repeats itself after some number of terms. If this happens, it is usually extremely important to solving the problem.

Example 2.1 (2012 AIME I #11)
Let $f_1(x) = \frac{2}{3} - \frac{3}{3x+1}$, and for $n \geq 2$, define $f_n(x) = f_1(f_{n-1}(x))$. The value of x that satisfies $f_{1001}(x) = x - 3$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. There's not much that immediately stands out about the function or equation that we are presented with. To get a better feel for the sequence, let's write out a few of the terms and see if we notice anything special. In general, when faced with a unfamiliar situation, it's a good problem solving strategy to look at some concrete examples and piece together data into observations and conjectures.

$$\begin{aligned} f_1(x) &= \frac{2}{3} - \frac{3}{3x+1}, \\ f_2(x) &= \frac{2}{3} - \frac{3}{3(\frac{2}{3} - \frac{3}{3x+1}) + 1} = \frac{2}{3} - \frac{3x+1}{3x-2}, \\ f_3(x) &= \frac{2}{3} - \frac{3}{3(\frac{2}{3} - \frac{3x+1}{3x-2}) + 1} = x. \end{aligned}$$

So the sequence is periodic! In particular, we see that $f_1(x), f_2(x), f_3(x), f_4(x), \dots$ repeats itself after every 3 terms. This means that $f_{1001}(x) = f_2(x)$. So we only need to solve

$$f_2(x) = x - 3 \iff \frac{2}{3} - \frac{3x+1}{3x-2} = x - 3$$

Solving this equation gives $x = \frac{5}{3}$ so the answer is $\boxed{8}$. □

Finally, another good strategy to try is generating functions. As we learned last class, we can often solve a summation by translating it into a generating function and using snake oil. Actually, the same method applies for sequences, as sometimes it is easier to first find the generating function, and backtrack to produce a general form for the sequence. Let's see this method in an example:

Example 2.2 (2018 PUMaC Algebra #7)
Let the sequence $\{a_n\}_{n=-2}^{\infty}$ satisfy $a_{-1} = a_{-2} = 0$, $a_0 = 1$, and for all non-negative integers n,

$$n^2 = \sum_{k=0}^{n} a_{n-k}a_{k-1} + \sum_{k=0}^{n} a_{n-k}a_{k-2}.$$

Given a_{2018} is rational, find the maximum integer m such that 2^m divide the denominator of the reduced form of a_{2018}.

Solution. At first sight, this reminds us a little of the Catalan recursion that we discussed in the previous handout. Therefore, we are going to use a similar strategy and try to determine a generating function for this sequence. Let $A(x) = a_0 + a_1x + a_2x^2 + \cdots$. Now, note that we can rewrite the equation we were given as

$$n^2 = \sum_{k=0}^{n} a_{n-k}(a_{k-1} + a_{k-2}).$$

The term $a_{k-1} + a_{k-2}$ seems hard to handle directly, so we are going to define a new sequence $b_k = a_{k-1} + a_{k-2}$. Let its generating function be $B(x)$. Note that this is defined for all nonnegative k due to the problem statement. Now, the equation in the problem becomes

$$n^2 = \sum_{k=0}^{n} a_{n-k}b_k.$$

Now, if we define a new sequence by $c_n = \sum_{k=0}^{n} a_{n-k}b_k$, then we have $c_n = n^2$ for all n. Let the generating function for this sequence be $C(x)$. Note that from the recursion we have $C(x) = A(x) \cdot B(x)$. This is because when we multiply $A(x)$ and $B(x)$, we obtain

$$\begin{aligned}(a_0 + a_1x^2 + a_2x^2 + \cdots)&(b_0 + b_1x + b_2x^2 + \cdots) \\ &= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots \\ &= c_0 + c_1x + c_2x^2 + \cdots \\ &= C(x)\end{aligned}$$

from the recursion for c_n. Now, we can find the the generating function for $C(x)$ because we know the closed form of c_n. We have $C(x) = \sum_{n=0}^{\infty} n^2x^n$. Now, the question might be how to obtain a term n^2 from the "known" generating functions. And the answer is right from the binomial coefficients! We already know the following generating functions:

$$\frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} 2\binom{n+2}{2}x^n = \sum_{n=0}^{\infty}(n^2+3n+2)x^n.$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \binom{n+1}{1}x^n = \sum_{n=0}^{\infty}(n+1)x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

We can combine these to find

$$C(x) = \sum_{n=0}^{\infty} n^2x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x} = \frac{x+x^2}{(1-x)^3} = A(x)B(x).$$

Now, our goal is to find $A(x)$, so we want to find a way to relate $A(x)$ and $B(x)$. Note that

$$B(x) = \sum_{n=0}^{\infty}(a_{n-1}+a_{n-2})x^n = x\sum_{n=0}^{\infty} a_nx^n + x^2\sum_{n=0}^{\infty} a_nx^n = (x+x^2)A(x)$$

Substituting,

$$(x+x^2)A(x)^2 = \frac{x+x^2}{(1-x)^3} \implies A(x) = \frac{1}{(1-x)^{\frac{3}{2}}}.$$

We claim that

$$\frac{1}{(1-x)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} \frac{2n+1}{4^n}\binom{2n}{n}x^n.$$

This can be proven either by taking a derivative of the Catalan generating function twice or expanding with the binomial theorem. Therefore, $a_{2018} = \frac{4037}{4^{2018}}\binom{4036}{2018}$. It suffices to compute

$$\nu_2\left(\binom{4036}{2018}\right) = \nu_2(4036!) - 2\nu_2(2018!) = 4029 - 2\cdot 2011 = 7.$$

There are 4036 powers of 2 in the denominator before dividing out the 7 powers of 2 from $\binom{4036}{2018}$, so there are $4036 - 7 = \boxed{4029}$ powers of 2 in the denominator.

□

In fact, this is a very general technique, and as we will see in the next section, it can be extended to a very broad class of recursions, namely linear recurrences.

§3 Linear Recursions

A linear recursion is one of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \; \forall n \geq k$$

for some constants $c_1, c_2, \ldots, c_k$. Generally, the most important thing when dealing with linear recursions is to find some sort of closed form for a_n given a set of initial conditions. Such a task can be achieved using last week's ideas on generating functions.

In particular, define the generating function

$$S = a_0 + a_1 x + a_2 x^2 + \ldots$$

What our recursion tells us is that

$$c_1 x S + c_2 x^2 S + c_2 x^2 S + \ldots + c_k x^k S = S - P(x)$$

where $P(x)$ is some polynomial with $\deg P < k$. To see this, simply consider the coefficent of x^n on the LHS for some $n \geq k$. Adding its coefficient over each term individually, we get $c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} = a_n$ where the equality is due to our recursive form.

This tells us that the coefficient of x^n on the LHS matches that of $P(x)$ for all $n \geq k$. However, we need $P(x)$ because this equality is not guaranteed for smaller coefficients, so $P(x)$ acts as a correcting factor.

Now that we have this equation, we can actually solve for S. In particular,

$$S = \frac{P(x)}{1 - c_1 x - c_2 x^2 - \ldots - c_k x^k}$$

Though this form may not seem useful. Note that the denominator can actually be factored! In particular, if we denote it as $Q(x)$, we can write it as

$$Q(x) = -c_k (x - r_1)(x - r_2) \cdots (x - r_k)$$

where the r_i are the roots of $Q(x)$ (for now, we assume that all the r_i are distinct.)

Now, using partial fraction decomposition, we know that we can break this fraction up into something of the form

$$S = \frac{m_1 r_1}{r_1 - x} + \frac{m_2 r_2}{r_2 - x} + \ldots + \frac{m_k r_k}{r_k - x}$$

Though we won't go into detail, the general idea for why this works is that all the polynomials of the form $\frac{\prod_{i=1}^{k}(x - r_i)}{x - r_j}$ are linearly independent, and thus form a basis over polynomials with degree $\leq k$. Thus, we are guaranteed to find such a partial fraction decomposition which will evaluate with numerator $-P(x)/c_k$.

Our last step, of course, is to note that $\frac{r_i}{x - r_i}$ is actually a generating function. In particular,

$$\frac{r_i}{r_i - x} = \frac{1}{1 - \frac{x}{r_i}} = \sum_{j=0}^{\infty} \left(\frac{x}{r_i} \right)^j$$

So, if we sum up across all i, we get that

$$S = \sum_{j=0}^{\infty} x^j \sum_{i=1}^{k} \frac{m_i}{r_i^j} \implies a_n = \sum_{i=1}^{k} \frac{m_i}{r_i^j}$$

In practice, to find such a closed form, we just need to find the values of $\frac{1}{r_i}$, and then use the first k terms of the recursion in order to calculate all the m_i.

Note that the $\frac{1}{r_i}$ are roots of the polynomial

$$C(x) = x^k - c_1 x^{k-1} - c_2 x^{k-2} - \ldots - c_k$$

This is known as the "characteristic polynomial" of the recursion.

Note that, if the roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ of $C(x)$ are distinct, we just get the aforementioned form, namely

$$a_n = m_1 \alpha_1^n + m_2 \alpha_2^n + \ldots + m_k \alpha_k^n$$

On the other hand, if we have a root with multiplicity, we must modify this form slightly. In particular, suppose α_1 actually has multiplicity two. Then, in our partial fraction decomposition, the numerator will be a linear expression rather than a constant, so we must replace m_1 with $a + bx$ for some a, b. So, the term corresponding to α_1 becomes $(a + bn)\alpha_1^n$. The idea is analogous for roots of higher multiplicity.

Now, let's try to apply the above ideas to find the closed form of a particular recursion:

Example 3.1 (Brilliant)
A sequence x_n is defined by $x_0 = -1$, $x_1 = 0$, $x_2 = 1$ and the recurrence relation

$$x_n = 6x_{n-1} - 12x_{n-2} + 8x_{n-3}.$$

Find the closed form of x_n.

Solution. Our first step should be to find the characteristic polynomial. We see that

$$C(x) = x^3 - 6x^2 + 12x - 8 = (x-2)^3$$

So the only root of this characteristic polynomial is -2, but this root has multiplicity 3. That means the closed form of our sequence is of the form

$$x_n = (an^2 + bn + c)2^n$$

for some constants a, b, c. Now we use the given values of x_0, x_1, x_2 to solve for a, b, c:

$$\begin{aligned} -1 &= x_0 = c, \\ 0 &= x_1 = 2(a + b + c), \\ 1 &= x_2 = 4(4a + 2b + c). \end{aligned}$$

Solving these equations gives $(a, b, c) = \left(-\frac{3}{8}, \frac{11}{8}, -1\right)$, so our closed form is

$$\boxed{x_n = \left(-\frac{3}{8}n^2 + \frac{11}{8}n - 1\right) 2^n}.$$

□

Example 3.2 (1990 AIME #15)
Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

Solution. We've done this problem before, through tricky algebraic manipulation. But let's see how our knowledge of linear recursions can solve this problem extremely quickly and efficiently!

Let $x_n = ax^n + by^n$; we're given the values of x_1, x_2, x_3, x_4, and we want to find x_5. What do we know about a sequence of the form $ax^n + by^n$? It must satisfy a linear recurrence relating x_n to x_{n-1} and x_{n-2}! In particular, we must have the recurrence

$$x_n = cx_{n-1} + dx_{n-2}$$

for some constants c and d. But we can solve for c and d with our given values. Choosing $n = 3$ and $n = 4$, we get the equations

$$16 = 7c + 3d, \ 42 = 16c + 7d.$$

Solving this system of linear equations gives $c = -14$ and $d = 38$, so

$$x_n = -14x_{n-1} + 38x_{n-2}.$$

Then $x_5 = -14x_4 + 38x_3 = -14 \cdot 42 + 38 \cdot 16 = \boxed{20}$. □

§4 Weird Sequences

Unfortunately, most sequences elude any specific categorization, and require an ad hoc idea in order to solve. Here are some general ideas in order to approach novel sequences:

- Try to compute small terms if possible. Guess a pattern through engineers induction and try to prove it
- Look at how the sequence behaves as a whole, and if there are any overarching global patterns
- Be on the look out for manipulations, such as factorizations and substitutions which will simplify how the sequence looks
- If initial conditions are given, see if they are special by trying the question with your own conditions. If something nice happens regardless of the values, the given information may be a decoy and you can use variables instead. Otherwise, try to understand why the numbers given are actually significant
- Try to prove subresults to get a better intuition with the sequence. Jot down ideas you may have or qualities of the sequence you see, even if they have little to do with what we are actually trying to prove

Often times, the recursive formula given may appear random or too chaotic. However, if how the sequence behaves as a whole is considered, then the solution becomes much more apparent. Let's see an example where, though the recursion isn't nice, there is still an underlying structure:

Example 4.1 (2017 CMIMC Algebra & Number Theory #9)

Define a sequence $\{a_n\}_{n=1}^{\infty}$ via $a_1 = 1$ and $a_{n+1} = a_n + \lfloor \sqrt{a_n} \rfloor$ for all $n \geq 1$. What is the smallest N such that $a_N > 2017$?

Solution. If one tries to immediately find a closed form for $\{a_i\}$, it is very easy to get lost. After all, $\lfloor \sqrt{x} \rfloor$ is already an unwieldy function, so trying to bash out terms by nesting it will make a complicated question even more complicated.

Instead, we try to see how $\lfloor \sqrt{x} \rfloor$ behaves. Suppose we have $a_k = m^2$ for some m. Then, $a_{k+1} = m^2 + m$, and $a_{k+2} = m^2 + 2m$. So, a_{k+2} is just under $(m+1)^2$. This means that, if we start from m^2, it will take three iterations to increment $\lfloor \sqrt{a_i} \rfloor$. However, if we start from a number between $m^2 + 1$ and $m^2 + m$, it will only take 2 increments. Furthermore, as we are only adding $2m$, if we started a away from m^2, we will be $a - 1$ away from $(m+1)^2$ after these two increments.

This gives us a very good idea about how the sequence will behave: At some point, a_n will overshoot a square, and produce a number a larger than a square. Then, every two turns the size of $\lfloor \sqrt{x} \rfloor$ will increase by 1, but the overshoot will decrease by 1, until it reaches 0 and we need three turns, which will in turn produce another overshoot.

Now, let's rigorize these ideas. We have that $a_1 = 1$ which is a square, so we need 3 iterations to get to $a_4 = 4$, which is another square. Three more iterations, we get to $a_7 = 10$, which now has overshoot of 1, so we can get to $a_9 = 16$ in two turns. Now, the next overshoot is over size 3, and it will take $2 * 3$ more turns to diminish back to a square, so we get $a_{9+3+2*3} = (4+4)^2 \implies a_{18} = 64$.

At this point, things are looking a bit fishy - in particular, we see that all perfect squares in our sequence are powers of 4. It is not hard to prove once we notice it - if $a_k = 4^n$, then after 3 turns we have an overshoot of $2^n - 1$, which will take $2 * (2^n - 1)$ turns to correct. This means $a_{k+3+2^{n+1}-2} = (2^n + 2^n)^2 \implies a_{k+2^{n+1}+1} = 4^{n+1}$. In fact, as we always add $2^{n+1} + 1$, this also gives us a closed form for k. See if you can show that $a_{2^{n+1}+n-1} = 4^n$.

The rest of the solution is surprisingly simple, especially as we now have so much intuition. We get that $a_{68} = 1024$, and we want to get to around $45^2 \approx 2017$. So, we need to increase the argument of our square by $45 - 32 = 13$, which will take $3 + 2 \cdot 12 = 27$ terms. After this many increments, our overshoot is $32 - 13 = 19$, so $a_{95} = 2025 + 19$. Now, $a_{94} = 2025 + 19 - 44 < 2017$, so our answer is $\boxed{95}$. □

Example 4.2

Romania TST 2003/1 Let $(a_n)_{n \geq 1}$ be a sequence for real numbers given by $a_1 = 1/2$ and for each positive integer n

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}.$$

Prove that for every positive integer n we have $a_1 + a_2 + \cdots + a_n < 1$.

Solution. At the beginning, it seems really unclear as to how to get the desired sum. So, rather than try to work backwards, we try to build up information about the sequence at hand.

First, note that if we take reciprocals, we get $\frac{1}{a_{n+1}} = 1 - \frac{1}{a_n} + \frac{1}{a_n^2}$, so setting $b_n = \frac{1}{a_n}$ gives the much nicer recurrence of

$$b_{n+1} = b_n^2 - b_n + 1$$

This is still not great. At the end of the day, we want to find a nice way to relate all the b_i. Let's try to list a few values to see what we get.

$b_1 = 2$, so $b_2 = 3$, $b_3 = 7$, $b_4 = 43$, $b_5 = 1807$, and so on. With these terms, it is not hard to guess that we must have $b_n = b_{n-1}b_{n-2} \dots b_1 + 1$. To prove, note that our recursion can be written as $\frac{b_{n+1}-1}{b_n-1} = b_n$, so if we multiply and telescope, we get

$$\frac{b_{n+1} - 1}{b_1 - 1} = b_1 b_2 \cdots b_n \implies b_{n+1} = b_1 b_2 \cdots b_n + 1$$

as desired. Note that we could have found this relation with the a_i as well, however first converting into b_i greatly reduced our computation for finding the pattern.

Now, let's convert back to as. We get that

$$a_{n+1} = \frac{1}{\frac{1}{a_1 \cdots a_n} + 1} = \frac{a_1 a_2 \cdots a_n}{1 + a_1 a_2 \cdots a_n} = (a_1 a_2 \dots a_n) - \frac{(a_1 a_2 \dots a_n)^2}{1 + a_1 a_2 \dots a_n}$$

$$= (a_1 a_2 \dots a_n) - (a_1 a_2 \dots a_{n+1})$$

Aha! Now, if we add all the as, we get a telescoping series. In particular,

$$a_1 + a_2 + \dots + a_n = 1 - a_1 a_2 \cdots a_n < 1$$

as desired. □

§5 Problems

Problem 5.1 (2018 AMC 12 #22)**.** Define a sequence recursively by $x_0 = 5$ and

$$x_{n+1} = \frac{x_n^2 + 5x_n + 4}{x_n + 6}$$

for all nonnegative integers n. Let m be the least positive integer such that

$$x_m \leq 4 + \frac{1}{2^{20}}.$$

In which of the following intervals does m lie?
(A) $[9, 26]$ **(B)** $[27, 80]$ **(C)** $[81, 242]$ **(D)** $[243, 728]$ **(E)** $[729, \infty]$

Problem 5.2 (Classic)**.** Derive a closed form for the Fibonacci numbers defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

Problem 5.3 (Brilliant)**.** A sequence x_n is defined by $x_0 = 0$, $x_1 = 1$, and the recurrence relation

$$x_n = x_{n-1} - 4x_{n-2}.$$

Find the closed form of x_n.

Problem 5.4. Prove that $\lfloor (2 + \sqrt{5}) \rfloor^{2020}$ is odd.

Problem 5.5 (2013 HMMT Algebra # 2)**.** let $\{a_n\}_{n\geq 1}$ be an arithmetic sequence $\{g_n\}_{n\geq 1}$ be a geometric sequence such that the first four terms of $\{a_n + g_n\}$ are 0,0, 1, and 0, in that order. What is the 10th term of $\{a_n + g_n\}$?

Problem 5.6 (1988 AIME # 2)**.** For any positive integer k, let $f_1(k)$ denote the square of the sum of the digits of k. For $n \geq 2$, let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1988}(11)$.

Problem 5.7 (2016 PUMaC Algebra #3)**.** Let $x_0, x_1, \ldots$ be a sequence of real numbers such that $x_n = \frac{1+x_{n-1}}{x_{n-2}}$ for $n \geq 2$. Find the number of ordered pairs of positive integers (x_0, x_1) such that the sequence gives $x_{2018} = \frac{1}{1000}$.

Problem 5.8 (2017 PUMaC Algebra #4)**.** Let $a_1, a_2, \ldots$ be a sequence of **positive** real numbers such that $a_n = 11a_{n-1} - n$ for all $n > 1$. The smallest possible value of a_1 can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problem 5.9 (2016 HMMT Algebra #5)**.** An infinite sequence of real numbers $a_1, a_2, \ldots$ satisfies the recurrence

$$a_{n+3} = a_{n+2} - 2a_{n+1} + a_n$$

for every positive integer n. Given that $a_1 = a_3 = 1$ and $a_{98} = a_{99}$, compute $a_1 + a_2 + \cdots + a_{100}$.

Problem 5.10 (Brilliant)**.** Let f be a function that satisfies the equation

$$(n+1)^2 f(n) - n^3 f(n-1) = 1$$

for all nonnegative integers n with $f(0) = 3$. Find $f(n)$.

Problem 5.11 (2016 PUMaC Algebra #4)**.** Define a sequence a_i as follows: $a_1 = 1$, $a_2 = 2015$, and $a_n = \frac{na_{n-1}^2}{a_{n-1} + na_{n-2}}$ for $n > 2$. What is the least k such that $a_k < a_{k-1}$?

Problem 5.12 (2009 AIME I #7)**.** The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1}-a_n)}-1 = \dfrac{1}{n+\frac{2}{3}}$ for $n \geq 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k.

Problem 5.13 (2016 AIME I #10)**.** A strictly increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ has the property that for every positive integer k, the subsequence $a_{2k-1}, a_{2k}, a_{2k+1}$ is geometric and the subsequence $a_{2k}, a_{2k+1}, a_{2k+2}$ is arithmetic. Suppose that $a_{13} = 2016$. Find a_1.

Problem 5.14 (2002 AIME I #12)**.** Let $F(z) = \frac{z+i}{z-i}$ for all complex numbers $z \neq i$, and let $z_n = F(z_{n-1})$ for all positive integers n. Given that $z_0 = \frac{1}{137} + i$ and $z_{2002} = a + bi$, where a and b are real numbers, find $a + b$.

Problem 5.15 (2009 AIME I #13)**.** The terms of the sequence (a_i) defined by $a_{n+2} = \frac{a_n+2009}{1+a_{n+1}}$ for $n \geq 1$ are positive integers. Find the minimum possible value of $a_1 + a_2$.

Problem 5.16 (2007 AIME I #14)**.** Let a sequence be defined as follows: $a_1 = 3$, $a_2 = 3$, and for $n \geq 2$, $a_{n+1}a_{n-1} = a_n^2 + 2007$. Find the largest integer less than or equal to $\frac{a_{2007}^2+a_{2006}^2}{a_{2007}a_{2006}}$.

Problem 5.17 (2009 AIME II # 14)**.** The sequence (a_n) satisfies $a_0 = 0$ and $a_{n+1} = \dfrac{8}{5}a_n + \dfrac{6}{5}\sqrt{4^n - a_n^2}$ for $n \geq 0$. Find the greatest integer less than or equal to a_{10}.

Problem 5.18 (2004 AIME I #15)**.** For all positive integers x, let

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ \frac{x}{10} & \text{if } x \text{ is divisible by } 10 \\ x+1 & \text{otherwise} \end{cases}$$

and define a sequence as follows: $x_1 = x$ and $x_{n+1} = f(x_n)$ for all positive integers n. Let $d(x)$ be the smallest n such that $x_n = 1$. (For example, $d(100) = 3$ and $d(87) = 7$.) Let m be the number of positive integers x such that $d(x) = 20$. Find the sum of the distinct prime factors of m.

Problem 5.19 (2014 AIME II #15)**.** For any integer $k \geq 1$, let $p(k)$ be the smallest prime which does not divide k. Define the integer function $X(k)$ to be the product of all primes less than $p(k)$ if $p(k) > 2$, and $X(k) = 1$ if $p(k) = 2$. Let $\{x_n\}$ be the sequence defined by $x_0 = 1$, and $x_{n+1}X(x_n) = x_n p(x_n)$ for $n \geq 0$. Find the smallest positive integer, t such that $x_t = 2090$.

Problem 5.20 (2019 PUMaC Algebra #7)**.** A doubly-index sequence $a_{m,n}$, for m and n nonnegative integers, is defined as follows.

(a) $a_{m,0} = 0$ for all $m > 0$ and $a_{0,0} = 1$.

(b) $a_{m,1} = 0$ for all $m > 1$, and $a_{1,1} = 1$, $a_{0,1} = 0$.

(c) $a_{0,n} = a_{0,n-1} + a_{0,n-2}$ for all $n \geq 2$.

(d) $a_{m,n} = a_{m,n-1} + a_{m,n-2} + a_{m-1,n-1} - a_{m-1,n-2}$ for all $m > 0$, $n \geq 2$.

Then there exists a unique value of x so $\sum\limits_{m=0}^{\infty}\sum\limits_{n=0}^{\infty} \frac{a_{m,n}x^m}{3^{m-n}} = 1$. Find $\lfloor 1000x^2 \rfloor$.

Problem 5.21 (USAMTS 30/3/5)**.** The sequence $\{a_n\}$ is defined by $a_0 = 1$, $a_1 = 2$, and for $n \geq 2$,

$$a_n = a_{n-1}^2 + (a_0 a_1 \cdots a_{n-2})^2.$$

Let k be a positive integer, and let p be a prime factor of a_k. Show that $p > 4(k-1)$.

Problem 5.22 (EGMO 2020/1)**.** The positive integers $a_0, a_1, a_2, \ldots, a_{3030}$ satisfy

$$2a_{n+2} = a_{n+1} + 4a_n \text{ for } n = 0, 1, 2, \ldots, 3028.$$

Prove that at least one of the numbers $a_0, a_1, a_2, \ldots, a_{3030}$ is divisible by 2^{2020}.

Problem 5.23 (IMO 2014/1)**.** Let $a_0 < a_1 < a_2 \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

Problem 5.24 (IMO 2018/2)**.** Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$ and

$$a_i a_{i+1} + 1 = a_{i+2},$$

for $i = 1, 2, \ldots, n$.