# Polynomials

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4/22

## §1 Introduction

**Definition 1.1.** A *polynomial* is a function in one variable of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $n$ is a nonnegative integer and the constants $a_0, a_1, \ldots a_n$ are the *coefficients* of the polynomial. If $a_n \neq 0$, then we say that the *degree* of the polynomial is $n$, and denote this by $\deg P$. We'll mostly consider polynomials where the coefficients are real numbers.

**Definition 1.2.** A complex number $z$ is a *root* of a polynomial $P$ if $P(z) = 0$.

An important fact about one-variable polynomials is:

**Theorem 1.3** (Fundamental Theorem of Algebra)
Any polynomial $P(x)$ with complex coefficients has at least one complex root, and has exactly $\deg P$ roots up to multiplicity.

**Definition 1.4.** We say that a polynomial $Q(x)$ is a *factor* of a polynomial $P(x)$ if there exists a polynomial $R(x)$ such that $P(x) = Q(x)R(x)$.

For example, $x + 1$ is a factor of $x^2 - 1$ (as $x^2 - 1 = (x - 1)(x + 1)$) but $x$ is not a factor of $x^2 - 1$. This is similar to the definition for integers: an integer $b$ is a factor of $a$ if there exists a third integer $c$ for which $a = bc$. The next theorem connects factors of a polynomial with its roots:

**Theorem 1.5** (Factor Theorem)
For a one-variable polynomial $P(x)$ and complex number $z$, $P(z) = 0$ if and only if $x - z$ is a factor of $P(x)$.

*Proof.* One direction is easy: if $x - z$ is a factor of $P(x)$, then

$$P(x) = (x - z)Q(x)$$

for some polynomial $Q$. Then

$$P(z) = (z - z)Q(z) = 0.$$

The converse is true as well: if $P(z) = 0$, then $x - z$ is a factor of $P(x)$. We'll see why this is true when we discuss the Remainder Theorem. □

Combining the Fundamental Theorem of Algebra and the Factor Theorem, we conclude the following important fact: if $z_1, z_2, \ldots, z_n$ are the roots of a polynomial $P$ (counted with multiplicity), then $P$ can be written in the form

$$P(x) = c(x - z_1)(x - z_2) \cdots (x - z_n)$$

for some constant $c$.

## §2 Vieta's Formulas

As we've seen, if $r_1, r_2, \ldots, r_n$ are the roots of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then we may write

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = a_n(x - r_1)(x - r_2) \cdots (x - r_n)$$

Let's expand the product of the $n$ linear factors on the right-hand side. Then we get

$$a_n x^n - a_n(r_1 + r_2 + \ldots + r_n)x^{n-1} + a_n(r_1r_2 + r_1r_3 + \ldots + r_{n-1}r_n) + \ldots + (-1)^n a_n r_1 r_2 \ldots r_n$$

where the coefficient of $x^k$ is $(-1)^{n-k}a_n$ times the sum of all possible products of $n-k$ of the roots. But we know these coefficients; they're precisely $a_0, a_1, \ldots, a_n$. Now if we set the coefficient of $x^k$ equal to $a^k$ we see that

$$\begin{aligned} r_1 + r_2 + \ldots + r_n &= -\frac{a_{n-1}}{a_n} \\ r_1r_2 + r_1r_3 + \ldots + r_{n-1}r_n &= \frac{a_{n-2}}{a_n} \\ &\cdots \\ r_1r_2 \cdots r_n &= (-1)^n \frac{a_0}{a_n} \end{aligned}$$

These equations relating the roots of the polynomial to its coefficients are called *Vieta's Formulas*, and can help use to compute various expressions involving the roots of a polynomial without having to compute the roots themselves.

**Example 2.1** (2008 AIME II # 7)

Let $r$, $s$, and $t$ be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

*Solution.* From Vieta's Formulas, we know the values

$$r + s + t = 0, \; rs + st + tr = \frac{1001}{8}, \; rst = -\frac{2008}{8} = -251.$$

From the first equation, we immediately see that $r + s = -t$ and $(r+s)^3 = -t^3$, so

$$(r+s)^3 + (s+t)^3 + (t+r)^3 = -(r^3 + s^3 + t^3).$$

Our goal is to express $r^3 + s^3 + t^3$ in terms of $r+s+t$, $rs+st+tr$, and $rst$, at which point we can substitute in numerical values and find the answer.

First, let's cube $r+s+t$ because we need $r^3+s^3+t^3$ to appear:

$$(r+s+t)^3 = r^3+s^3+t^3+3(r^2s+s^2t+t^2r+rs^2+st^2+tr^2)+6rst.$$

Now we need to get rid of all of the terms appearing after $r^3+s^3+t^3$. To get rid of $3r^2s$, we can try subtracting $3(r+s+t)(rs+st+tr)$ (because this contains a $3r^2s$ term):

$$3(r+s+t)(rs+sr+tr) = 3(r^2s+s^2t+t^2r+rs^2+st^2+tr^2)+9rst.$$

So now we have

$$(r+s+t)^3 - 3(r+s+t)(rs+st+tr) = r^3+s^3+t^3-3rst.$$

Now all we have to do is add $3rst$ to both sides:

$$r^3+s^3+t^3 = (r+s+t)^3 - 3(r+s+t)(rs+st+tr)+3rst.$$

From here, we just have to plug in the numbers. Since $r+s+t=0$, most of the terms vanish:

$$-r^3-s^3-t^3 = -3rst = (-3)(-251) = \boxed{753}.$$

Alternatively, you may be aware of the factorization

$$r^3+s^3+t^3-3rst = \frac{1}{2}(r+s+t)((r-s)^2+(s-t)^2+(t-r)^2)$$

which makes it much easier to express $r^3+s^3+t^3$ in terms of $r+s+t, rs+st+tr$, and $rst$. □

**Example 2.2** (2013 HMMT Algebra # 5)

Let $a$ and $b$ be real numbers, and let $r$,$s$, and $t$ be the roots of $f(x)=x^3+ax^2+bx-1$. Also, $g(x)=x^3+mx^2+nx+p$ has roots $r^2$,$s^2$, and $t^2$. if $g(-1)=-5$, find the maximum possible value of $b$.

*Solution.* From Vieta's formulas, we know that

$$-a = r+s+t,\ \ b = rs+st+tr,\ \ 1 = rst$$

and

$$m = -(r^2+s^2+t^2),\ \ n = r^2s^2+s^2t^2+t^2r^2,\ \ p = -r^2s^2t^2.$$

Let's try to express $m, n$, and $p$ in terms of $a$ and $b$. The easiest one is $p:$

$$p = -(rst)^2 = -1.$$

For $m$, we square $r+s+t:$

$$a^2 = (r+s+t)^2 = r^2+s^2+t^2+2rs+2st+2tr = -m+2b \implies m = 2b-a^2.$$

Finaly, for $n$ we can square $rs+st+tr:$

$$b^2 = (rs+st+tr)^2 = r^2s^2+s^2t^2+t^2r^2+2r^2st+2rs^2t+2rst^2 = n+2rst(r+s+t) = n-2a \implies n = b^2+2a.$$

So now we write $g(x)$ in terms of $a$ and $b:$

$$g(x) = x^3+(2b-a^2)x^2+(b^2+2a)x-1.$$

We know that $g(-1) = -5$; when we plug this into our equation for $g(x)$ we get

$$-5 = (-1)^3 + (2b - a^2)(-1)^2 + (b^2 + 2a)(-1) - 1 = 2b - a^2 - b^2 - 2a - 2 \implies a^2 + 2a + (b^2 - 2b - 3) = 0.$$

We seek the largest possible value of $b$; since $a$ is real, we know that the discriminant of this quadratic must be nonnegative. In particular,

$$2^2 - 4(b^2 - 2b - 3) \geq 0 \implies b^2 - 2b - 4 \leq 0.$$

Solving this quadratic gives us that the largest possible value of $b$ is $\boxed{1 + \sqrt{5}}$.

A second solution is possible by factoring $f(x)$ and $g(x)$; try the same approach as in Example 3.2. □

## §3 Factorization

If we have an expression of the form $P(x) = Q(x) \cdot \text{something}$, then we immediately know that $Q(x)$ is a factor of $P(x)$. As we will see, this can be very useful information.

**Example 3.1** (2016 AIME I # 11)

Let $P(x)$ be a nonzero polynomial such that $(x-1)P(x+1) = (x+2)P(x)$ for every real $x$, and $(P(2))^2 = P(3)$. Then $P\left(\frac{7}{2}\right) = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

*Solution.* Let's look at what linear polynomials are factors of $P(x)$. Since $x - 1$ appears on the left hand side, it must be a factor of the right hand side. So we immediately know that $x - 1$ is a factor of $(x+2)P(x)$, and so it is a factor of $P(x)$. Similarly, $x + 2$ appears on the right hand side, so it must appear on the left. Thus, we know that $x + 2$ is a factor of $P(x + 1)$, or that $x + 1$ is a factor of $P(x)$. So we may write

$$P(x) = (x-1)(x+1)Q(x)$$

for some polynomial $Q(x)$. Let's plug back into the original equation:

$$(x-1)x(x+2)Q(x+1) = (x+2)(x-1)(x+1)Q(x) \implies xQ(x+1) = (x+1)Q(x).$$

Note that $x$ appears on the left hand side, so $x$ must be a factor of $Q(x)$. As $x+1$ appears on the right hand side, it must be a factor of $Q(x + 1)$, but this also implies that $x$ is a factor of $Q(x)$. Now we can write

$$Q(x) = xR(x)$$

for a polynomial $R(x)$. So we have

$$x(x+1)R(x) = (x+1)(x)R(x) \implies R(x) = R(x+1).$$

Now we know that $R$ is constant! One way to see this is to see that

$$\cdots = R(-2) = R(-1) = R(0) = R(1) = R(2) = \cdots,$$

which implies that $R(x) - R(0)$ has a root at every integer and hence is the zero polynomial. Thus $R(x) = R(0)$ for all real $x$. Suppose that $R(x)$ is the constant polynomial $c$; then we get

$$Q(x) = xR(x) = cx, \;\; P(x) = (x-1)(x+1)Q(x) = cx(x-1)(x+1).$$

In order to find $c$, we can use the given numerical information about $P(2)$ and $P(3)$ :

$$P(2) = c \cdot 2 \cdot 1 \cdot 3 = 6c, \;\; P(3) = c \cdot 3 \cdot 2 \cdot 4 = 24c.$$

Since $(P(2))^2 = P(3)$, we have

$$(6c)^2 = 24c \implies 36c^2 = 24c \implies c = \frac{2}{3}$$

(we know that $c \neq 0$ because otherwise $P$ would be the zero polynomial). Now we can calculate $P\left(\frac{7}{2}\right)$ :

$$P\left(\frac{7}{2}\right) = \frac{2}{3} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{9}{2} = \frac{105}{4}$$

and the answer is $105 + 4 = \boxed{109}$. □

It can often be useful to consider the factorization of a polynomial given by the Fundamental Theorem of Algebra:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = a_n(x - r_1)(x - r_2)\ldots(x - r_n).$$

Thus, if we are faced with expressions similar to $(z - r_1)(z - r_2)\cdots(z - r_n)$, we can try to relate them to $P(z)$. For instance,

$$(1 - r_1)(1 - r_2)\ldots(1 - r_n) = \frac{P(1)}{a_n}$$

**Example 3.2** (2014 USAMO/1)

Let $a$, $b$, $c$, $d$ be real numbers such that $b - d \geq 5$ and all zeros $x_1, x_2, x_3$, and $x_4$ of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

*Solution.* The condition $b - d \geq 5$ is strange; it seems a bit arbitrary and disconnected from what we wish to find. In these situations, it's often a good idea to first ignore these conditions and see how they can be used later on.

We wish to find $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$. We can use the same Vieta methods as we did for Example 2.2, but with four variables this is much bashier and more painful. Let's try a different approach.

We know that

$$P(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

so if we factor the difference of squares, we get

$$\begin{aligned}
&(x - x_1^2)(x - x_2^2)(x - x_3^2)(x - x_4^2) \\
&= (\sqrt{x} - x_1)(\sqrt{x} - x_2)(\sqrt{x} - x_3)(\sqrt{x} - x_4)(\sqrt{x} + x_1)(\sqrt{x} + x_2)(\sqrt{x} + x_3)(\sqrt{x} + x_4) \\
&= P(\sqrt{x})P(-\sqrt{x}).
\end{aligned}$$

Let's try this on our given expression:

$$\begin{aligned}
(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1) &= (x_1^2 - i^2)(x_2^2 - i^2)(x_3^2 - i^2)(x_4^2 - i^2) \\
&= (x_1 - i)(x_2 - i)(x_3 - i)(x_4 - i)(x_1 + i)(x_2 + i)(x_3 + i)(x_4 + i) \\
&= P(i)P(-i) \\
&= (1 - ai - b + ci + d)(1 + ai - b - ci + d) \\
&= (1 - b + d + (c - a)i)(1 - b + d - (c - a)i) \\
&= (b - d - 1)^2 + (c - a)^2
\end{aligned}$$

Now it's clear where the condition $b - d \geq 5$ comes in: we have

$$(b-d-1)^2 + (c-a)^2 \geq (5-1)^2 + 0^2 = 16.$$

But we're not done yet; we need to exhibit a polynomial $P(x)$ that achieves the value 16. For this, we need $b - d = 5$ and $c - a = 0$ to hold. Fortunately, there's an easy polynomial that satisfies this condition:

$$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$$

and we conclude that the answer is indeed $\boxed{16}$. □

## §4 Remainder Theorem

The Remainder Theorem is a very useful generalization of the Factor Theorem. While the result itself can be powerful, the ideas behind its proof are far more applicable to a wide variety of problems. First, a definition:

**Definition 4.1.** The *remainder* of $P(x)$ when it is divided by $Q(x)$ is the unique polynomial $R(x)$ with $\deg R < \deg Q$ such that $Q(x)$ is a factor of $P(x) - R(x)$. In other words, if we write

$$P(x) = Q(x)S(x) + R(x)$$

for polynomials $R(x)$ and $S(x)$ with $\deg R < \deg Q$, then the remainder is $R(x)$. For instance, since $x^2 - x + 1 = x(x-1) + 1$, the remainder when $x^2 - x + 1$ is divided by $x$ is 1.

**Theorem 4.2** (Remainder Theorem)
The remainder when $P(x)$ is divided by $x - c$ is $P(c)$.

*Proof.* Write

$$P(x) = (x-c)Q(x) + R(x),$$

where $R(x)$ is our desired remainder. Since $\deg R < \deg\,(x-c) = 1$ we see that $R$ must be a constant polynomial. Suppose we plug in $x = c$ into the above equation:

$$P(c) = (c-c)Q(c) + R(c) = R(c).$$

Since $R$ is a constant polynomial, we see that $R(x) = P(c)$ for all $x$. □

Notice that this immediately implies the Factor Theorem: $x - c$ is a factor of $P(x)$ if and only if the remainder $R(x) = 0$, which occurs if and only if $P(c) = 0$. The ideas behind this proof (writing out the divison and remainder and plugging in suitable values) allow us to tackle much more than the remainder upon division by linear polynomials.

**Example 4.3** (AoPS Intermediate Algebra)
Find the remainder when $x^{100} - 4x^{98} + 5x + 6$ is divided by $x^3 - 2x^2 - x + 2$.

*Solution.* Since the degree of $x^3 - 2x^2 - x + 2$ is 3, the remainder has degree at most 2, so we can express it as $ax^2 + bx + c$ for constants $a, b, c$. Let's write out the division:

$$x^{100} - 4x^{98} + 5x + 6 = (x^3 - 2x^2 - x + 2)Q(x) + ax^2 + bx + c.$$

However, we note that we can actually factor $x^3 - 2x^2 - x + 2$ :

$$x^{100} - 4x^{98} + 5x + 6 = (x - 2)(x - 1)(x + 1)Q(x) + ax^2 + bx + c.$$

As in the proof of the Remainder Theorem, we substitute in $x = 2, 1, -1$, which makes the term containing $Q(x)$ equal to zero:

$$\begin{aligned} 4a + 2b + c &= 2^{100} - 4 \cdot 2^{98} + 5 \cdot 2 + 6 = 16 \\ a + b + c &= 1^{100} - 4 \cdot 1^{98} + 5 \cdot 1 + 6 = 8 \\ a - b + c &= (-1)^{100} - 4 \cdot (-1)^{98} + 5(-1) + 6 = -2. \end{aligned}$$

We can solve this system of linear equations to get $(a, b, c) = (1, 5, 2)$, so the remainder is $\boxed{x^2 + 5x + 2}$. □

**Example 4.4** (2016 PUMaC Algebra A # 4)

Suppose that $P$ is a polynomial with integer coefficients such that $P(1) = 2, P(2) = 3$ and $P(3) = 2016$. If $N$ is the smallest possible positive value of $P(2016)$, find the remainder when $N$ is divided by 2016.

*Solution.* Consider what happens when we divide $P(x)$ by $(x - 1)(x - 2)$. We get

$$P(x) = (x - 1)(x - 2)Q(x) + R(x),$$

where $R(x)$ has degree 1. Then, if we plug in $x = 1$ and $x = 2$, we get $R(1) = 2$ and $R(2) = 3$. Since $R$ is linear, we have that $R(x) = x + 1$. Therefore,

$$P(x) = (x - 1)(x - 2)Q(x) + x + 1.$$

If we plug in $x = 3$, then we obtain $2016 = 2Q(3) + 4$, which yields $Q(3) = 1006$. Now, if we plugged in $x = 2016$, then we get $P(2016) = 2015 \cdot 2014Q(2016) + 2017$. To finish, we will utilize the following lemma.

**Lemma 4.5**

If $P(x)$ is a polynomial with integer coefficients, and $a, b$ are integers with $a \neq b$, then $(a - b)|(P(a) - P(b))$.

*Proof.* Let $P(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$. Then,

$$P(a) - P(b) = c_n(a^n - b^n) + c_{n-1}(a^{n-1} - b^{n-1}) + \cdots + c_1(a - b).$$

Each term in this sum is divisible by $a - b$, and since $P$ has integer coefficients, $P(a) - P(b)$ is divisible by $a - b$. □

Now, we can use the lemma to obtain $2013|(Q(2016) - Q(3))$. The minimum value of $Q(2016)$ such that $P(2016) \geq 0$ is $Q(2016) = Q(3) = 1006$. With this value, we get

$$P(2016) = 2015 \cdot 2014 \cdot 1006 + 2017 \equiv (-1)(-2)(1006) + 2017 \equiv \boxed{2013} \pmod{2016}.$$

□

Notice that in these two problems, we never cited the Remainder Theorem; rather, we used and generalized the ideas behind its proof.

## §4.1 Aside on Lagrange Interpolation

There are many possible problems of the form, "$P(x)$ is the polynomial of least degree so that $P(x) =$ something for $x = 1, 2, \dots, n$. Find $P(n+1)$." We'll give an example of the general strategy that typically works on problems like these, then discuss a messier alternative.

> **Example 4.6** (2017 HMMT Algebra and Number Theory #6)
> A polynomial $P$ of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \dots, 2016$. Find $\lfloor 2017P(2017) \rfloor$.

*Solution.* We know that $P(x) - \frac{1}{x^2}$ is equal to zero at $x = 1, 2, \dots, 2016$, so we'd like to say that this polynomial is divisible by $(x-1)(x-2)\cdots(x-2016)$. But we can't, because $P(x) - \frac{1}{x^2}$ isn't a polynomial! Fortunately, there's an easy fix:

$$P(x) - \frac{1}{x^2} = 0 \implies x^2P(x) - 1 = 0,$$

so $x^2P(x) - 1$ is a polynomial divisible by $(x-1)(x-2)\cdots(x-2016)$. Since $P$ has degree 2015, $x^2P(x) - 1$ has degree 2017. Thus we may write

$$x^2P(x) - 1 = c(x-1)(x-2)\cdots(x-2016)(x-k)$$

for some constants $c, k$. In other words,

$$x^2P(x) = c(x-1)(x-2)\cdots(x-2016)(x-k) + 1$$

Thus the right-hand side must be divisible by $x^2$, so it constant and linear coefficients must both be zero. If we expand the right-hand side, then the constant term is

$$c(-1)(-2)(-2016)(-k) + 1 = 1 - 2016!ck$$

and the linear coefficient is

$$c \cdot 2016! - ck((-2)(-3)\cdots(-2016) + (-1)(-3)\cdots(-2016) + \dots(-1)(-2)\cdots(-2015))$$

$$= c \cdot 2016! + ck \cdot 2016! \left(1 + \frac{1}{2} + \cdots + \frac{1}{2016}\right) = 2016!c(1 + kH_{2016})$$

where we let $H_{2016} = 1 + \frac{1}{2} + \cdots + \frac{1}{2016}$ for convenience. Now we set the linear coefficient to zero:

$$2016!c(1 + kH_{2016}) = 0 \implies k = -\frac{1}{H_{2016}}$$

and the constant coefficient to zero:

$$1 - 2016!ck = 0 \implies c = \frac{1}{2016!k} = -\frac{H_{2016}}{2016!}$$

Thus we may write

$$x^2P(x) = 1 - \frac{H_{2016}}{2016!}(x-1)(x-2)\cdots(x-2016)(x + \frac{1}{H_{2016}})$$

and we get

$$2017^2P(2017) = 1 - \frac{H_{2016}}{2016!} \cdot 2016! \cdot (2017 + \frac{1}{H_{2016}}) = -2017H_{2016}$$

so the desired value is $2017P(2017) = -H_{2016}$. We leave the approximation $\lfloor -H_{2016} \rfloor = \boxed{-9}$ to the interested reader. □

There is actually a general formula for the polynomial in problems of this form. We state it below.

**Theorem 4.7** (Lagrange Interpolation Formula)

Let $a_1, a_2, \dots a_n, b_1, b_2, \dots b_n$ be real numbers. The unique polynomial $P$ of degree $\leq n-1$ such that

$$P(a_1) = b_1, P(a_2) = b_2, \dots P(a_n) = b_n$$

is

$$P(x) = \sum_{i=1}^{n} \frac{(x-a_1)(x-a_2)\cdots(x-a_{i-1})(x-a_{i+1})\cdots(x-a_n)}{(a_i-a_1)(a_i-a_2)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)} b_i$$

For instance, in the polynomial from example 4.4, the unique polynomial of degree $\leq 2$ satisfying $P(1) = 2, P(2) = 3, P(3) = 2016$ is

$$P(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} \cdot 2 + \frac{(x-1)(x-3)}{(2-1)(2-3)} \cdot 3 + \frac{(x-1)(x-2)}{(3-1)(3-2)} \cdot 2016.$$

From this example, you may be able to see the structure behind the formula. The $i^{th}$ term of the sum is equal to $b_i$ when $x = a_i$ and is equal to zero when $x$ is equal to any other $a_j$. This way, when we sum all $n$ terms, we get a polynomial that takes the value $b_i$ at every $x = a_i$. The strategy we used to solve Example 4.5 is usually the much cleaner approach, but sometimes it is not viable. In these cases, we must either use trickier algebraic manipulations or bash with Lagrange Interpolation.

## §5 Quadratics

Like any other polynomial, we can express a quadratic in terms of its roots $r, s$:

$$ax^2 + bx + c = a(x-r)(x-s).$$

However, quadratics possess a unique property: they are *symmetric*. Thus we have another method of representing a quadratic, its *vertex form*:

$$ax^2 + bx + c = a(x-h)^2 + k$$

where $(h, k)$ is the *vertex* of the quadratic. These ways of representing quadratics can come in handy in various circumstances.

**Example 5.1** (2020 AIME I # 14)

Let $P(x)$ be a quadratic polynomial with complex coefficients whose $x^2$ coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a+b)^2$.

*Solution.* Since the $x^2$ coefficient of $P(x)$ is 1, if $r$ and $s$ are the two roots of $P(x)$ then

$$P(x) = (x-r)(x-s).$$

If $P(P(x)) = 0$ then we must have either $P(x) = r$ or $P(x) = s$. Each of these quadratics have exactly two solutions, so we know that two of $P(3), P(4), P(a), P(b)$ are equal to $r$

and the other two are equal to $s$. We consider two different cases:

**Case 1.** $P(3) = P(4)$. Without loss of generality, let $P(3) = P(4) = r$; then we have that $a$ and $b$ are the two solutions to $P(x) = s$. In other words, 3 and 4 are the solutions to

$$(x-r)(x-s) = r \implies x^2 - (r+s)x + (rs-r) = 0$$

while $a$ and $b$ are the solutions to

$$(x-r)(x-s) = s \implies x^2 - (r+s)x + (rs-s) = 0.$$

By Vieta on the second equation, we see that $a+b = r+s$. But by Vieta on the first equation, we have that $r+s = 3+4 = 7$. So in this case, we have $(a+b)^2 = 7^2 = 49$.

**Case 2.** $P(3) \neq P(4)$. In this case, let $P(3) = r$ and $P(4) = s$, and assume that $P(a) = r$ and $P(b) = s$. Then 3 and $a$ are the solutions to

$$(x-r)(x-s) = r \implies x^2 - (r+s)x + (rs-r) = 0$$

so by Vieta $3 + a = r + s$ and $a = r+s-3$. Similarly, 4 and $b$ are the solutions to

$$(x-r)(x-s) = s \implies x^2 - (r+s)x + (rs-s) = 0.$$

so by Vieta $4 + b = r+s$ and $b = r+s-4$. Thus $a+b = 2r+2s-7$, so all we need is to find $r+s$. Let's write out the equations $P(3) = r$ and $P(4) = s$ :

$$(3-r)(3-s) = r \implies rs - 3r - 3s + 9 = r,$$
$$(4-r)(4-s) = s \implies rs - 4r - 4s + 16 = s.$$

When we subtract the first equation from the second equation, we can actually solve for $s$!

$$r - s = (rs - 3r - 3s + 9) - (rs - 4r - 4s + 16) = r + s - 7 \implies s = \frac{7}{2}.$$

Now we can substitute this value of $s$ into either of our two original equations to solve for $r$ :

$$(4-r)\left(4 - \frac{7}{2}\right) = \frac{7}{2} \implies r = -3.$$

Thus we have

$$a + b = 2r + 2s - 7 = 2 \cdot \frac{7}{2} + 2(-3) - 7 = -6$$

so $(a+b)^2 = 36$ in this case.

Finally, the requested sum is $49 + 36 = \boxed{85}$.

$\square$

**Example 5.2** (ARML 2017 Tiebreaker # 1)
Compute the least positive $N$ such that there exists a quadratic polynomial $f(x)$ with integer coefficients satisfying

$$f(f(1)) = f(f(5)) = f(f(7)) = f(f(11)) = N.$$

*Solution.* The first thing to note is that the given inputs are symmetric around $x = 6$. Since a quadratic has an axis of symmetry, this suggests that $x = 6$ is our axis of symmetry. We can confirm this as follows: If $f(x)$ has an axis of symmetry around $x = n$, then $f(f(x))$ does as well. Indeed, if $f(x)$ is symmetric around $x = n$ then

$$f(x) = f(2n - x) \implies f(f(x)) = f(f(2n - x))$$

which is equivalent to $f(f(x))$ being symmetric around $x = n$. Now $f(f(x))$ is a quartic, so $f(f(x)) - N$ has exactly four roots. Clearly, $f(f(x)) - N$ is also symmetric around $x = n$, so these four roots are symmetric about $x = n$. But these four roots are $1, 5, 7, 11$, and the only possible value for the value of $n$ is 6.

Since $f(x)$ has axis of symmetry $x = n$, we can write $f(x)$ in vertex form:

$$f(x) = a(x - 6)^2 + k$$

for nonzero $a$. If we substitute in $x = 1, 5, 7, 11$, we obtain

$$f(5) = f(7) = a + k,\ f(1) = f(11) = 25a + k.$$

So now we know that $f(a + k) = f(25a + k) = N$. But $a + k \neq 25a + k$, so the only way we can have $f(a + k) = f(25a + k)$ is if they are symmetric about 6; that is,

$$a + k = 2 \cdot 6 - (25a + k) \implies k = 6 - 13a.$$

Thus we have

$$f(x) = a(x - 6)^2 + 6 - 13a.$$

Now we want to compute $N = f(f(1))$ we see that

$$f(1) = 6 + 12a \implies N = f(f(1)) = 144a^3 - 13a + 6.$$

Since $f(x) = a(x - 6)^2 + 6 - 13a$ must have integer coefficients, we see that $a$ must be a nonzero integer. Thus we want the smallest positive value of $144a^3 - 13a + 6$ over all nonzero integers. We leave it as an exercise to check that this is minimized at $a = 1$, for an answer of $144 - 13 + 6 = \boxed{137}$. □

# §6 Problems

**Problem 6.1** (2017 AMC 12A #23)**.** For certain real numbers $a$, $b$, and $c$, the polynomial $g(x) = x^3 + ax^2 + x + 10$ has three distinct roots, and each root of $g(x)$ is also a root of the polynomial $f(x) = x^4 + x^3 + bx^2 + 100x + c$. What is $f(1)$?

**Problem 6.2** (PUMaC 2016 Algebra #2)**.** Let $f(x) = 15x - 2016$. If $f(f(f(f(f(x))))) = f(x)$, find the sum of all possible values of $x$.

**Problem 6.3** (CMIMC 2018 Algebra #3)**.** Let $P(x) = x^2 + 4x + 1$. What is the product of all real solutions to the equation $P(P(x)) = 0$?

**Problem 6.4** (1996 AIME #5)**.** Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are $a, b$, and $c$, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b, b + c$, and $c + a$. Find $t$.

**Problem 6.5** (2001 AIME I #3)**.** Find the sum of the roots, real and non-real, of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$, given that there are no multiple roots.

**Problem 6.6** (2010 AMC 12A # 21)**.** The graph of $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2$ lies above the line $y = bx + c$ except at three values of $x$, where the graph and the line intersect. What is the largest of those values?

**Problem 6.7** (2010 AMC 12B # 23)**.** Monic quadratic polynomials $P(x)$ and $Q(x)$ have the property that $P(Q(x))$ has zeroes at $x = -23, -21, -17, \text{and} -15$, and $Q(P(x))$ has zeroes at $x = -59, -57, -51, \text{and} -49$. What is the sum of the minimum values of $P(x)$ and $Q(x)$?

**Problem 6.8** (2014 HMMT Algebra #4)**.** Let $b$ and $c$ be real numbers, and define the polynomial $P(x) = x^2 + bx + c$. Suppose that $P(P(1)) = P(P(2)) = 0$, and that $P(1) \neq P(2)$. Find $P(0)$.

**Problem 6.9** (Purple Comet 2010 #25)**.** Let $x_1, x_2$, and $x_3$ be the roots of the polynomial $x^3 + 3x + 1$. There are relatively prime positive integers $m$ and $n$ such that

$$\frac{m}{n} = \frac{x_1^2}{(5x_2+1)(5x_3+1)} + \frac{x_2^2}{(5x_1+1)(5x_3+1)} + \frac{x_3^2}{(5x_1+1)(5x_2+1)}.$$

Find $m + n$.

**Problem 6.10** (2010 AIME I #6)**.** Let $P(x)$ be a quadratic polynomial with real coefficients satisfying

$$x^2 - 2x + 2 \leq P(x) \leq 2x^2 - 4x + 3$$

for all real numbers $x$, and suppose $P(11) = 181$. Find $P(16)$.

**Problem 6.11** (2007 AIME I # 8)**.** The polynomial $P(x)$ is cubic. What is the largest value of $k$ for which the polynomials $Q_1(x) = x^2 + (k - 29)x - k$ and $Q_2(x) = 2x^2 + (2k - 43)x + k$ are both factors of $P(x)$?

**Problem 6.12** (2015 AIME I #10)**.** Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

**Problem 6.13** (USAMO 1984 #1)**.** The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is $-32$. Determine the value of $k$.

**Problem 6.14.** Find the remainder when $x^{28} + 1$ is divided by $x^4 + x^3 + x^2 + x + 1$.

**Problem 6.15** (2016 PUMaC Algebra A # 7)**.** Let $S_P$ be the set of all polynomials $P$ with complex coefficients, such that $P(x^2) = P(x)P(x-1)$ for all complex numbers $x$. Suppose $P_0$ is the polynomial in $S_P$ of maximal degree such that $P_0(1)|2016$. Find $P_0(10)$.

**Problem 6.16** (2020 HMMT Algebra and Number Theory #8)**.** Let $P(x)$ be the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, ..., 2020$. Compute $P(2021^2)$.

**Problem 6.17** (1984 AIME 1 # 15)**.** Determine $w^2 + x^2 + y^2 + z^2$ if

$$\frac{x^2}{2^2-1} + \frac{y^2}{2^2-3^2} + \frac{z^2}{2^2-5^2} + \frac{w^2}{2^2-7^2} = 1$$
$$\frac{x^2}{4^2-1} + \frac{y^2}{4^2-3^2} + \frac{z^2}{4^2-5^2} + \frac{w^2}{4^2-7^2} = 1$$
$$\frac{x^2}{6^2-1} + \frac{y^2}{6^2-3^2} + \frac{z^2}{6^2-5^2} + \frac{w^2}{6^2-7^2} = 1$$
$$\frac{x^2}{8^2-1} + \frac{y^2}{8^2-3^2} + \frac{z^2}{8^2-5^2} + \frac{w^2}{8^2-7^2} = 1.$$

**Problem 6.18** (IMO Shortlist 2005/N3)**.** Let $a$, $b$, $c$, $d$, $e$, $f$ be positive integers and let $S = a + b + c + d + e + f$. Suppose that the number $S$ divides $abc + def$ and $ab + bc + ca - de - ef - df$. Prove that $S$ is composite.