Geometry B

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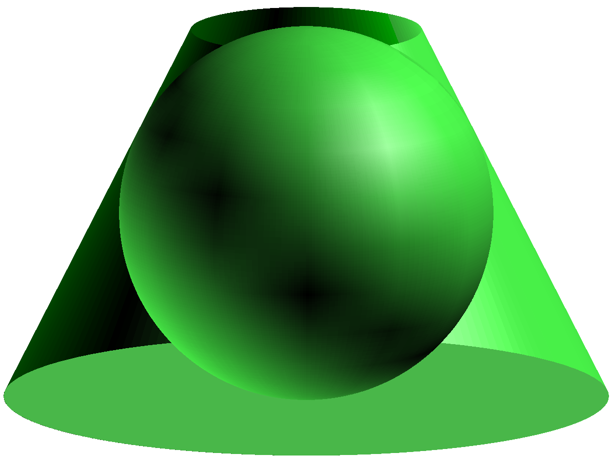
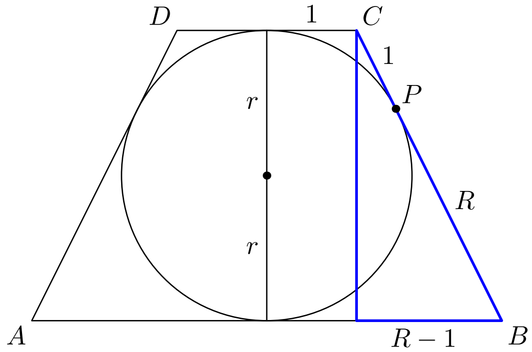
§1 3D Geometry

Almost every 3-dimensional geometry problem boils doing to using the right projection or cross section to convert the problem into a 2-dimensional problem that could be solved with standard geometry techniques such as similar triangles, Pythagorean theorem, etc. Some problems may require the use of multiple cross sections.

Example 1.1 (2014 AMC 12B #19)
A sphere is inscribed in a truncated right circular cone as shown. The volume of the truncated cone is twice that of the sphere. What is the ratio of the radius of the bottom base of the truncated cone to the radius of the top base of the truncated cone?



Solution. It is not immediately clear how to find the ratio of the radii from the three-dimensional diagram, so we will try to take a cross section. The only cross section that makes sense to take here (one that preserves essential features of the diagram) is the vertical cross section. See the following diagram:



Without loss of generality, let's assume that the top base has radius 1 and that the radius of the sphere is r. Also, let's assume the bottom base has radius R. Note that we want to find the value of R.

In this cross section, the circle is tangent to all four sides of $ABCD$. Therefore, since tangents from a point to a circle are equal, $BP = R$ Similarly, we have $CP = 1$. Therefore, we can form the blue triangle as marked with legs $2r$ and $R - 1$, and hypotenuse $R + 1$. By the Pythagorean Theorem,

$$(2r)^2 + (R - 1)^2 = (R + 1)^2 \implies R = r^2.$$

Now that we can express the dimensions of both solids in terms of r, we can use the volume condition to solve for the value of r. To find the volume of the truncated cone, we will use the following fact:

Lemma 1.2

The volume of a truncated cone or frustum with radii r_1 and r_2, and height h is $\frac{\pi}{3}h(r_2^2 + r_1r_2 + r_1^2)$.

Proof. We can consider the volume to be the difference between the volume of the initial cone and the volume of the small cone that was removed. Both cones are similar and have radii r_1 and r_2, respectively. If h_1 is the height of the small cone and h_2 is the height of the large cone, similar triangles gives $h_2 = \frac{r_2}{r_1}h_1$ and $h_2 - h_1 = h$. Therefore,

$$h_1\left(\frac{r_2}{r_1} - 1\right) = h \implies h_1 = \frac{hr_1}{r_2 - r_1}.$$

We get that $h_2 = \frac{hr_2}{r_2 - r_1}$. Now, we can compute the volume:

$$\frac{1}{3}\pi r_2^2 \cdot \frac{hr_2}{r_2 - r_1} - \frac{1}{3}\pi r_1^2 \cdot \frac{hr_1}{r_2 - r_1} = \frac{\pi}{3}h \cdot \frac{r_2^2 - r_1^3}{r_2 - r_1} = \frac{\pi}{3}h(r_2^2 + r_1r_2 + r_1^2)$$

□

Now, the volume of the truncated cone is $\frac{\pi}{3}(2r)(r^4 + r^2 + 1)$, and the volume of the sphere is $\frac{4\pi}{3}r^3$. Since the truncated cone has twice the volume of the sphere, we get

$$\frac{\pi}{3}(2r)(r^4 + r^2 + 1) = \frac{8\pi}{3}r^3$$

$$r^4 + r^2 + 1 = 4r^2$$

Substituting $R = r^2$, we get $R^2 + R + 1 = 4R$ or $R^2 - 3R + 1 = 0$. Once we solve the quadratic, we get $R = \boxed{\dfrac{3+\sqrt{5}}{2}}$.

□

Example 1.3 (2015 PUMaC Geometry A #4)

Find the largest r such that 4 balls each of radius r can be packed into a regular tetrahedron with side length 1. In a packing, each ball lies outside every other ball, and every ball lies inside the boundaries of the tetrahedron. If r can be expressed in the form $\frac{\sqrt{a}+b}{c}$ where $a, b, c,$ are integers such that $\gcd(b, c) = 1$, what is $a + b + c$?

Solution. Note that the centers of the balls form a tetrahedron of side-length $2r$. We now note that the distance from the center of the tetrahedron to a face of the larger one is r more than the distance from the center of the tetrahedron to a face of the smaller one.

Lemma 1.4

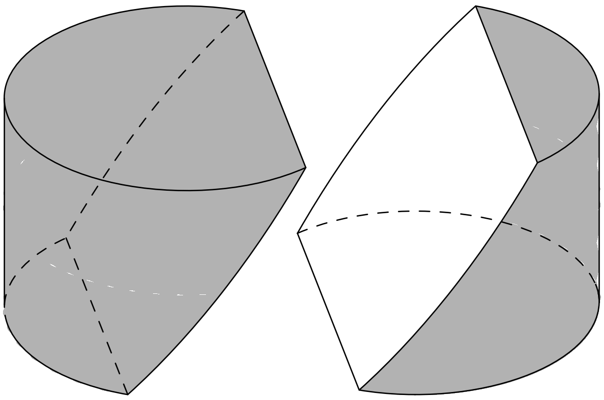
The distance from the center of a regular tetrahedron to any of its faces is $\frac{\sqrt{6}}{12}s$, where s is the side-length of the tetrahedron.

Proof. We do this by calculating the volume of the tetrahedron in 2 different ways. First, we consider it as a single pyramid. Note that it has a base area of $\frac{\sqrt{3}}{4}s^2$. Note that if we drop an altitude, it has length $\sqrt{1-\frac{1}{3}}s$, since it forms a right triangle with hypotenuse s and other leg $\frac{s}{\sqrt{3}}$. Thus, we find that the volume of the tetrahedron is $\frac{1}{3}\left(\frac{\sqrt{3}}{4}s^2\right)\left(\sqrt{\frac{2}{3}}s\right) = \frac{\sqrt{2}}{12}s^3$. Now, we note that we may also consider it as splitting into 4 smaller congruent tetrahedrons, each with base one of the 4 original faces and vertex at the center of the tetrahedron. Then, if the distance from the center of the tetrahedron to a face is d, we have the volume is $4\left(\frac{1}{3}\left(\frac{\sqrt{3}}{4}s^2\right)d\right) = \frac{\sqrt{3}}{3}s^2 d$. Thus, setting these two expressions equal gives $\frac{\sqrt{2}}{12}s^3 = \frac{\sqrt{3}}{3}s^2 d$, so $d = \frac{\sqrt{6}}{12}s$, as desired. □

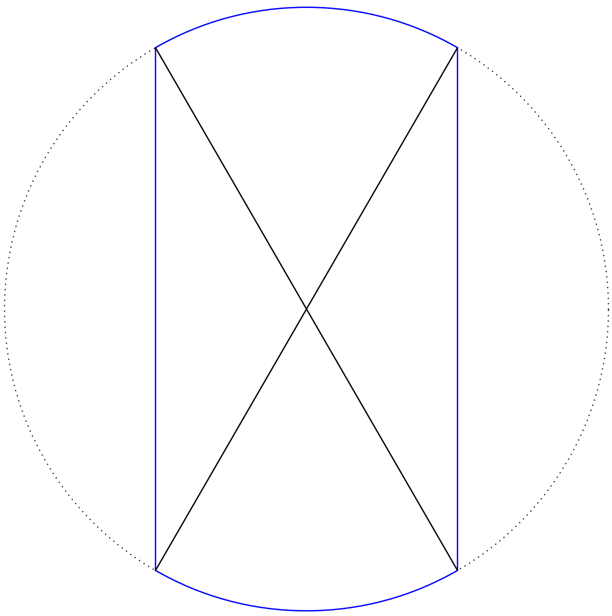
Now, note that we have the distance from the center of the tetrahedron to a face of the larger tetrahedron is $\frac{\sqrt{6}}{12}$ while the distance from the center of the tetrahedron to a face of the smaller one is $\frac{\sqrt{6}}{6}r$, so we find that we have the equation $\frac{\sqrt{6}}{12} = r + \frac{\sqrt{6}}{6}r$. Thus, solving for r gives $r = \frac{\frac{\sqrt{6}}{12}}{1+\frac{\sqrt{6}}{6}} = \frac{\sqrt{6}-1}{10}$. Thus, we find our answer to be $6 + (-1) + 10 = \boxed{15}$ □

Example 1.5 (2015 AIME I #15)

A block of wood has the shape of a right circular cylinder with radius 6 and height 8, and its entire surface has been painted blue. Points A and B are chosen on the edge on one of the circular faces of the cylinder so that arc AB on that face measures $120°$. The block is then sliced in half along the plane that passes through point A, point B, and the center of the cylinder, revealing a flat, unpainted face on each half. The area of one of those unpainted faces is $a \cdot \pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a + b + c$.



Solution. Consider a projection of the area of the unpainted face onto the base of the cylinder.



Now, the area of this projection is much easier to compute. We can write it as the sum of two circular sectors with angle 60° and two triangles as shown in the diagram. Now, the area of the projection is

$$2 \cdot \frac{1}{6}(36\pi) + 2 \cdot \frac{1}{2} \cdot 3 \cdot 6\sqrt{3} = 12\pi + 18\sqrt{3}.$$

In this figure, the width between the two vertical lines is 6. However, in the original unpainted face, there is a distance of $\sqrt{6^2 + 8^2} = 10$. Therefore, the unpainted face is formed by stretching the projection by $\frac{10}{6} = \frac{5}{3}$. Finally, the area of the unpainted face is

$$\frac{5}{3} \cdot (12\pi + 18\sqrt{3}) = 20\pi + 30\sqrt{3}.$$

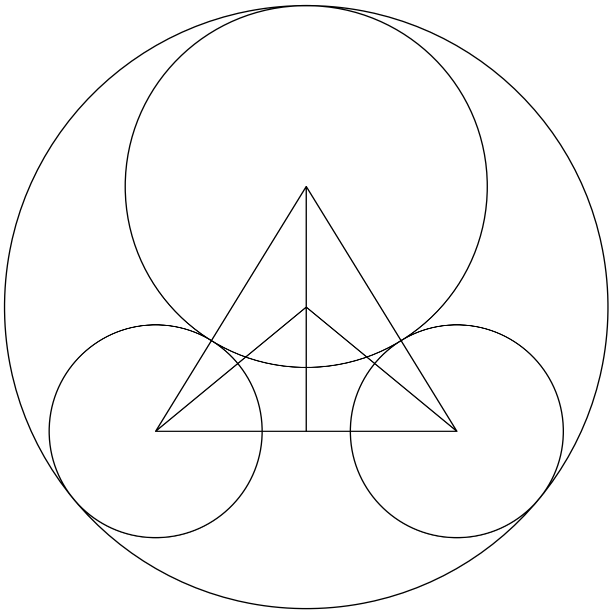
The answer is $20 + 30 + 3 = \boxed{053}$.

□

Example 1.6 (2020 Purple Comet Math Meet #30)

Four small spheres each with radius 6 are each internally tangent to a larger sphere with radius 17. The four small spheres form a ring with each of the four sphere externally tangent to its two neighboring small spheres. A sixth intermediately sized sphere is internally tangent to the large sphere and externally tangent to each of the four small spheres. Its radius is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. We note that the centers of the four small spheres must form a square. We now wish to take a cross section that preserves as many of the tangencies as we can. Thus, we take one through the centers of both the large and intermediate spheres, as well as two opposite centers of the small spheres. Our diagram now looks like the below:



We note that, because the centers of the two small spheres form a diagonal in a square of side 12, they must be $12\sqrt{2}$ apart. Now, suppose the radius of the intermediate circle is r. Notice that the distance between the centers of the large and intermediate circles is $17-r$, the distance between the centers of the large and small circles is 11, and the distance between the centers of the intermediate and small circles is $6+r$. Now, with 2 applications of the Pythagorean Theorem, we may find that $(6\sqrt{2})^2+\left((17-r)+\sqrt{11^2-(6\sqrt{2})^2}\right)^2 = (6+r)^2$. The LHS simplifies into $72+(24-r)^2 = r^2-48r+648$ while the RHS simplifies into $r^2+12r+36$. Thus, setting these equal gives $r = \frac{612}{60} = \frac{51}{5}$. Thus, our answer is $51+5 = \boxed{56}$. □

§2 Geometric Optimization

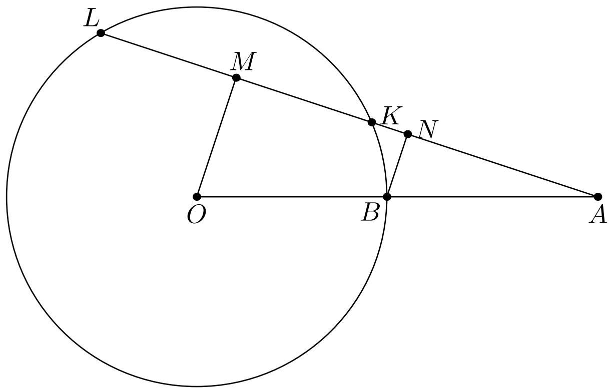
Many problems asking us to maximize or minimize a certain geometric quantity can be approached in the same way as other geometry problems: we compute the lengths, areas, angles, and other relevant quantities in the figure in terms of a variable x. Once we can express the desired quantity in terms of x, we can then use algebraic methods to determine the optimal value.

Other times, however, the diagram can become very complicated, and the quantities we wish to optimize can be quite difficult to compute directly. In these cases, it's important to *make geometric simplifications first* and reduce the given configuration into something simpler.

In general, if there is no clear, simple way to directly calculate the given quantity, then it is a good idea to try to make synthetic observations until the calculations become clear. (This is a good rule of thumb for all geometry problems, not just optimization.) In most of the examples below, we'll need to make some geometric observations before jumping into the computation.

Example 2.1 (2013 AIME II # 10)

Given a circle of radius $\sqrt{13}$, let A be a point at a distance $4+\sqrt{13}$ from the center O of the circle. Let B be the point on the circle nearest to point A. A line passing through the point A intersects the circle at points K and L. The maximum possible area for $\triangle BKL$ can be written in the form $\frac{a-b\sqrt{c}}{d}$, where a, b, c, and d are positive integers, a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.



Solution. Let's begin by writing an expression to calculate the area of $\triangle BKL$. If N is the foot of the altitude from B to $\overline{AL}$, then

$$[BKL] = \frac{1}{2} BN \cdot KL.$$

So we will need to determine BN and KL. However, we can simplify the problem a lot with the following observation: let M be the foot of the altitude from O to $\overline{AL}$. Then $\overline{OM} \parallel \overline{BN}$, so

$$\frac{OM}{AO} = \frac{BN}{AB} \implies BN = \frac{AB}{AO} OM = \frac{4}{4+\sqrt{13}} OM.$$

Thus

$$[BKL] = \frac{1}{2} \cdot \frac{4}{4+\sqrt{13}} OM \cdot KL = \frac{4}{4+\sqrt{13}} [OKL].$$

So we just need to maximize $[OKL]$. We can use a different formula for this:

$$[OKL] = \frac{1}{2} OL \cdot OK \cdot \sin \angle KOL = \frac{13}{2} \sin \angle KOL$$

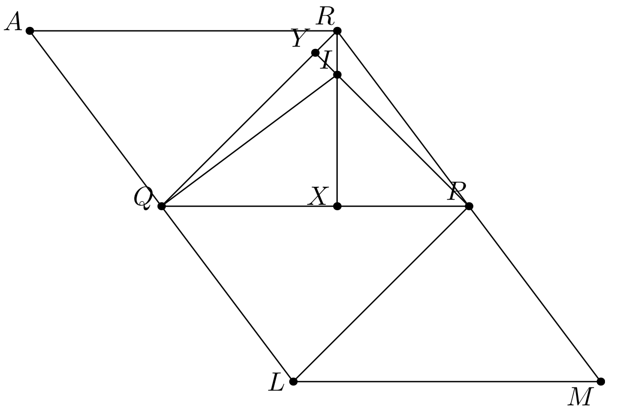
since OL and OK are radii of the given circle. Now this is clearly optimized when $\angle KOL = 90^\circ$, since the sine is at most 1. This is also achievable; we can draw a situation in which $\angle KOL = 90^\circ$. Thus the largest possible area is

$$\frac{4}{4+\sqrt{13}} \cdot \frac{13}{2} = \frac{104-26\sqrt{13}}{3}$$

and the answer is $104 + 26 + 13 + 3 = \boxed{146}$. $\square$

Example 2.2 (2019 ARML Individual #8)

In parallelogram $ARML$, points P and Q are the midpoints of sides $\overline{RM}$ and $\overline{AL}$, respectively. Point X lies on segment $\overline{PQ}$, and $PX = 3$, $RX = 4$, and $PR = 5$. Point I lies on segment $\overline{RX}$ such that $IA = IL$. Compute the maximum possible value of $\frac{[PQR]}{[LIP]}$.



Solution. The diagram we are given is quite complicated and the points forming the triangles are seemingly unrelated, making direct computation very difficult. Let's see what we can discover about the configuration.

First, we see that $\angle PXR = 90^\circ$ by the given lengths (we have a $3-4-5$ right triangle), so $\overline{RI} \perp \overline{PQ}$. We can also discover another perpendicularity involving I: since $IA = IL$ and $QA = QL$ we have $\overline{IQ} \perp \overline{AL}$. But $ARML$ is a parallelogram so this implies $\overline{IQ} \perp \overline{RP}$. Hence I is actually the orthocenter of triangle PQR.

This lets us handle triangle LIP, which was previously difficult because the three points had seemingly no relations between them. The orthocenter allows us to deduce the third perpendicularity $\overline{IP} \perp \overline{QR}$. But we see that $\overline{QR} \parallel \overline{LP}$ (since $ARML$ is a parallelogram and P, Q are midpoints of opposite sides). Thus $\overline{IP} \perp \overline{LP}$. So triangle LIP is actually right!

Now we can efficiently handle the area of triangle LIP:

$$[LIP] = \frac{1}{2} LP \cdot IP = \frac{1}{2} IP \cdot QR.$$

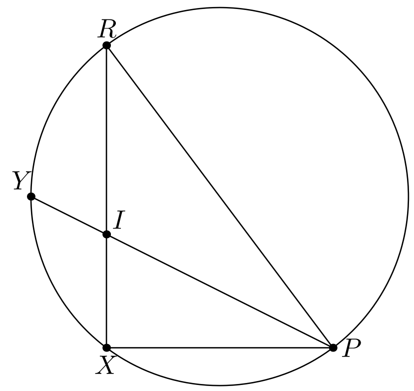
If $\overline{PI}$ meets $\overline{QR}$ at Y, then

$$[PQR] = \frac{1}{2} QR \cdot PY$$

so

$$\frac{[PQR]}{[LIP]} = \frac{PY \cdot QR}{PI \cdot QR} = \frac{PY}{PI}.$$

If we look closely, we see that $\angle PXR = \angle PYR = 90^\circ$, meaning that P, R, X, Y all lie on a circle with diameter $\overline{PR}$. We now have the following picture:



We have a $3-4-5$ triangle PXR, and we select a point I on segment RX. We extend $\overline{PI}$ to meet the circumcircle of triangle PXR again at Y, and we wish to maximize the ratio $\frac{PY}{PI}$.

With the geometry of the problem greatly simplified, we can now begin the computation. Let $\theta = \angle IPX$. Then from right triangle PXI we get $PI = \frac{3}{\cos\theta}$. On the other hand, the extended law of sines on triangle PRY gives $PY = 5\sin\angle PRY$. We compute

$$\angle PRY = \angle PRX + \angle XRY = \angle PRX + \theta$$

which implies $\sin\angle PRY = \frac{4}{5}\sin\theta + \frac{3}{5}\cos\theta$. Thus our desired ratio is

$$\frac{PY}{PI} = \frac{4\sin\theta + 3\cos\theta}{\frac{3}{\cos\theta}} = \frac{4}{3}\sin\theta\cos\theta + \cos^2\theta = \frac{2}{3}\sin 2\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2}.$$

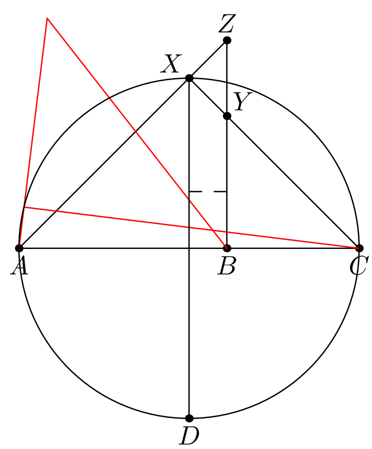
We leave it as an exercise to check that the largest possible value of this expression is

$$\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2} + \frac{1}{2} = \boxed{\frac{4}{3}}.$$

□

Example 2.3 (2016 MPfG #19)

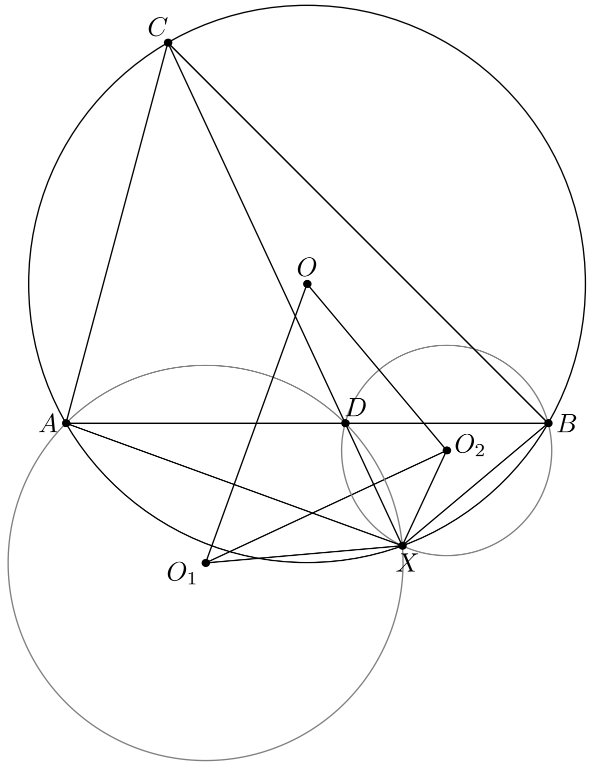
In the coordinate plane, consider points $A = (0,0)$, $B = (11,0)$, and $C = (18,0)$. Line ℓ_A has slope 1 and passes through A. Line ℓ_B is vertical and passes through B. Line ℓ_C has slope -1 and passes through C. The three lines ℓ_A, ℓ_B, ℓ_C begin rotating clockwise about points A, B, and C, respectively. They rotate at the same angular rate. At any given time, the three lines form a triangle. Determine the largest possible area of such a triangle.



Solution. We note that the lines will always form a 45-45-90 triangle, since the angles between the lines is constant. Now, note that the area of the triangle is then determined by the distance from the intersection of ℓ_A and ℓ_C to line ℓ_B. Call the intersection of ℓ_A and ℓ_C X. Now, note that we have that $\angle AXC = 90$ for all X, so X must lie on the circle with diameter AC. Now, let D be the point $(9, -9)$, which is the midpoint of one of the arcs AC. Now, note that $\measuredangle DXA = \measuredangle CXD = 45$, so we find that DX is always parallel to ℓ_B. Thus, the distance from X to ℓ_B is just the distance from D to ℓ_B. However, note that this distance is always at most the length of BD, which is $\sqrt{85}$. Because the area of the triangle is equal to the square of the distance from X to ℓ_B, we have that the area is at most $\sqrt{85}^2 = \boxed{85}$. □

Example 2.4 (USOMO 2020/1)

Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.



Solution. As the radical axis of two circles is always perpendicular to the line passing through the two centers, we have $OO_1 \perp AX$, $O_1O_2 \perp DX$. So, $\angle OO_1O_2$ is a 90 degree rotation of $\angle AXC$, which has measure $\angle B$. Similarly, $\angle O_1O_2O = \angle BAC$, so $OO_1O_2 \sim CBA$.

Now, it suffices to minimize O_1O_2. Note that the projection of O_1 onto AB is the midpoint of AD, and that of O_2 is the midpoint of BD, so the projection of line O_1O_2 onto AB has length half of AB. Hence, $O_1O_2 \geq \frac{AB}{2}$. In fact, this is achievable if and only if $O_1O_2 \parallel AB \implies CX \perp AB$, so X is the unique point such that CX is perpendicular to AB. □

§3 Problems

Problem 3.1 (2016 AIME I #4)**.** A pyramid has a triangular base with side lengths 20, 20, and 24. The three edges of the pyramid from the three corners of the base to the fourth vertex of the pyramid all have length 25. The volume of the pyramid is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

Problem 3.2 (2019 CSMC #6)**.** Suppose that $ABCD$ is a square with side length 4 and that $0<k<4$. Let points P, Q, R, and S be on BC, CD, DA, and AP, respectively, so that

$$\frac{BP}{PC}=\frac{CQ}{QD}=\frac{DR}{RA}=\frac{AS}{SP}=\frac{k}{4-k}.$$

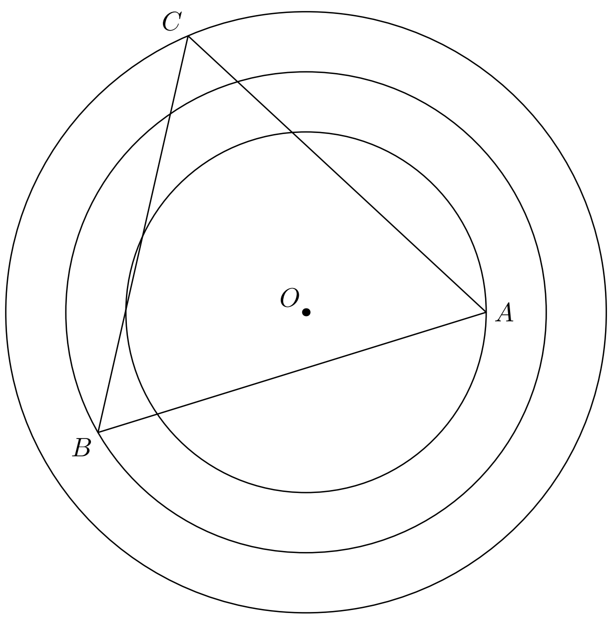
What is the value of k which minimizes the area of quadrilateral $PQRS$?

Problem 3.3 (2020 AOIME #7)**.** Two congruent right circular cones each with base radius 3 and height 8 have axes of symmetry that intersect at right angles at a point in the interior of the cones a distance 3 from the base of each cone. A sphere with radius r lies inside both cones. The maximum possible value for r^2 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 3.4 (2020 AIME I #6)**.** A flat board has a circular hole with radius 1 and a circular hole with radius 2 such that the distance between the centers of the two holes is 7. Two spheres with equal radii sit in the two holes such that the spheres are tangent to each other. The square of the radius of the spheres is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 3.5 (USAMTS 3/1/31)**.** Circle ω is inscribed in unit square $PLUM$, and points I and E lie on ω such that U, I, and E are collinear. Find, with proof, the greatest possible area for $\triangle PIE$.

Problem 3.6 (2014 AIME II #9)**.** A cylindrical barrel with radius 4 feet and height 10 feet is full of water. A solid cube with side length 8 feet is set into the barrel so that the diagonal of the cube is vertical. The volume of water thus displaced is v cubic feet. Find v^2.

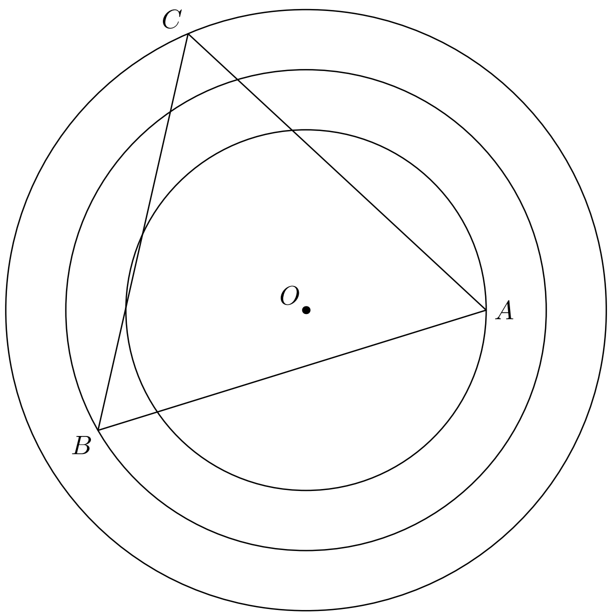


Problem 3.7 (2016 AIME I #9)**.** Triangle ABC has $AB = 40, AC = 31$, and $\sin A = \frac{1}{5}$. This triangle is inscribed in rectangle $AQRS$ with B on $\overline{QR}$ and C on $\overline{RS}$. Find the maximum possible area of $AQRS$.

Problem 3.8 (2017 Purple Comet Math Meet #27)**.** A container the shape of a pyramid has a 12×12 square base, and the other four edges each have length 11. The container is partially filled with liquid so that when one of its triangular faces is lying on a flat surface, the level of the liquid is half the distance from the surface to the top edge of the container. Find the volume of the liquid in the container.

Problem 3.9 (2016 PUMaC Geometry A #3)**.** Let C be a right circular cone with apex A. Let P_1, P_2, P_3, P_4 and P_5 be points placed evenly along the circular base in that order, so that $P_1P_2P_3P_4P_5$ is a regular pentagon. Suppose that the shortest path from P_1 to P_3 along the curved surface of the cone passes through the midpoint of AP_2. Let h be the height of C, and r be the radius of the circular base of C. If $\left(\frac{h}{r}\right)^2$ can be written in simplest form as $\frac{a}{b}$, find $a + b$.

Problem 3.10 (2012 AIME I # 13)**.** Three concentric circles have radii 3, 4, and 5. An equilateral triangle with one vertex on each circle has side length s. The largest possible area of the triangle can be written as $a + \frac{b}{c}\sqrt{d}$, where a, b, c and d are positive integers, b and c are relatively prime, and d is not divisible by the square of any prime. Find $a + b + c + d$.



Problem 3.11 (2019 Purple Comet Math Meet #29)**.** In a right circular cone, A is the vertex, B is the center of the base, and C is a point on the circumference of the base with $BC = 1$ and $AB = 4$. There is a trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$. A right circular cylinder whose surface contains the points A, C¡ and D intersects the cone such that its axis of symmetry is perpendicular to the plane of the trapezoid, and CD is a diameter of the cylinder. A sphere of radius r lies inside the cone and inside the cylinder. The greatest possible value of r is $\frac{a\sqrt{b}-c}{d}$, where a, b, c, and d are positive integers, a and d are relatively prime, and b is not divisible by the square of any prime. Find $a + b + c + d$.

Problem 3.12 (2017 Purple Comet Math Meet #30)**.** A container is shaped like a right circular cone with base diameter 18 and height 12. The vertex of the container is pointing down, and the container is open at the top. Four spheres, each with radius 3, are placed inside the container as shown. The first sphere sits at the bottom and is tangent to the cone along a circle. The second, third, and fourth spheres are placed so they are each tangent to the cone and tangent to the first sphere, and the second and fourth spheres are each tangent to the third sphere. The volume of the tetrahedron whose vertices are at the centers of the spheres is K. Find K^2

Problem 3.13 (2018 Purple Comet Math Meet #30)**.** One right pyramid has a base that is a regular hexagon with side length 1, and the height of the pyramid is 8. Two other right pyramids have bases that are regular hexagons with side length 4, and the heights of those pyramids are both 7. The three pyramids sit on a plane so that their bases are adjacent to each other and meet at a single common vertex. A sphere with radius 4 rests above the plane supported by these three pyramids. The distance that the center of the sphere is from the plane can be written as $\frac{p\sqrt{q}}{r}$, where p, q, and r are relatively prime positive integers, and q is not divisible by the square of any prime. Find $p+q+r$.

Problem 3.14 (2008 AIME I # 14)**.** Let $\overline{AB}$ be a diameter of circle ω. Extend $\overline{AB}$ through A to C. Point T lies on ω so that line CT is tangent to ω. Point P is the foot of the perpendicular from A to line CT. Suppose $AB = 18$, and let m denote the maximum possible length of segment BP. Find m^2.

Problem 3.15 (2009 AIME II # 15)**.** Let $\overline{MN}$ be a diameter of a circle with diameter 1. Let A and B be points on one of the semicircular arcs determined by $\overline{MN}$ such that A is the midpoint of the semicircle and $MB = \frac{3}{5}$. Point C lies on the other semicircular arc. Let d be the length of the line segment whose endpoints are the intersections of diameter $\overline{MN}$ with the chords $\overline{AC}$ and $\overline{BC}$. The largest possible value of d can be written in the form $r - s\sqrt{t}$, where r, s, and t are positive integers and t is not divisible by the square of any prime. Find $r+s+t$.

Problem 3.16 (2018 AIME I # 13)**.** Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

Problem 3.17. The pyramid $SABCD$ has parallelogram $ABCD$ as base and vertex S. Let the midpoint of edge SC be P. Consider plane $AMPN$ where M is on edge SB and N is on edge SD. Find the maximum value of r_1 and maximum value r_2 of $\frac{V_1}{V_2}$ where V_1 is the volume of pyramid $SAMPN$ and V_2 is the volume of pyramid $SABCD$. Express your answer as an ordered pair (r_1, r_2).

Problem 3.18 (2017 AIME II #15)**.** Tetrahedron $ABCD$ has $AD = BC = 28$, $AC = BD = 44$, and $AB = CD = 52$. For any point X in space, define $f(X) = AX + BX + CX + DX$. The least possible value of $f(X)$ can be expressed as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

Problem 3.19 (Sharygin 2020/24)**.** Let I be the incenter of a tetrahedron $ABCD$, and J be the center of the exsphere touching the face BCD containing three remaining faces (outside these faces). The segment IJ meets the circumsphere of the tetrahedron at point K. Which of two segments IK and JK is longer?

Problem 3.20 (Germany 2020/6)**.** Given tetrahedron $ABCD$, consider its insphere ω and D-exsphere Ω. If ω touches ABC at X and Ω touches it at Y, show $\angle BAX = \angle CAY$