Geometry A

Adithya B., Brian L., William W., Daniel X.

7/1

§1 Basic Geometry

To start, we will cover geometry problems that require some elementary techniques. We first review some basic concepts, such as similar triangles and power of a point.

Definition 1.1. Triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if any of the following hold:

- $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$
- $\angle CAB = \angle C'A'B'$, $\angle ABC = \angle A'B'C'$, or any cyclic variation

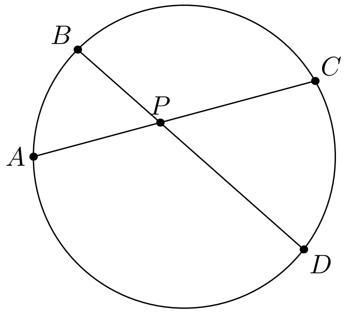
Of course, if one of the bullets hold, then the other one does as well. In fact, if two triangles are similar, we can generalize the first bullet point to state $\frac{f(ABC)}{f(A'B'C')} = \frac{AB}{A'B'}$ where f denotes any linear function of the triangle. This can include, for example, the length of the A-altitude, the circumradius, the inradius, etc.

Often times, the key idea of a geometry question is to find an important pair of similar triangles, as demonstrated by the following:

Theorem 1.2 (Power of a Point)
If A, B, C, D are on a circle, and AC, BD intersect at P, then

$$AP \cdot CP = BP \cdot DP$$

Proof. We only consider the case where P lies inside the circle (try to use a similar method to prove the other cases!)



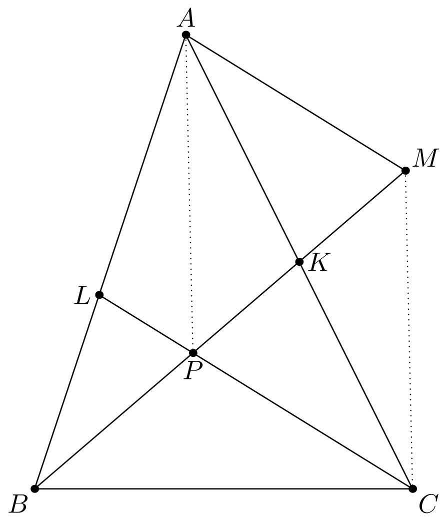
Note that $\angle BAC, \angle BDC$ point to the same arc, so $\angle BAP = \angle PDC$. On the other hand, $\angle BPA, \angle CPD$ are vertical angles so $\angle BPA = \angle CPD$. Hence, $BAP \sim CDP$.

Now, this means we have $\frac{BP}{AP} = \frac{CP}{DP} \implies AP \cdot CP = BP \cdot DP$, as desired. □

Of course, these aren't the only tools we will need to solve early geometry questions, but they should be sufficient for most questions. Other important, common ideas are noticing parallelograms and noting that triangles with the same base and same height have the same area. We will see both ideas in action in the following questions.

Example 1.3 (2009 AIME I #5)

Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.



Solution. Note that $AK = CK$, $PK = MK$, and $\angle AKM = \angle CKP$, so we see $\triangle AKM \cong \triangle CKP$. In particular, we see that $AMCP$ is a parallelogram. To finish, we will use the angle bisector theorem. By the angle bisector theorem,

$$\frac{AL}{LB} = \frac{AC}{BC} = \frac{3}{2}.$$

Also, since $LP \parallel AM$, $\triangle BLP \sim \triangle BAM$, we have

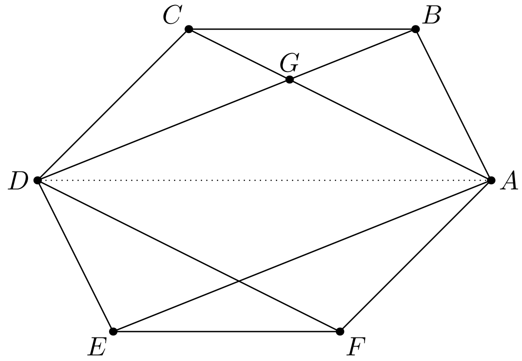
$$\frac{LP}{AM} = \frac{BL}{BA} = \frac{1}{1 + \frac{LA}{LB}} = \frac{2}{5}$$

Thus, $LP = \frac{2}{5}AM = \boxed{072}$.

□

Example 1.4 (2019 HMMT Geometry # 4)

Convex hexagon $ABCDEF$ is drawn in the plane such that $ACDF$ and $ABDE$ are parallelograms with area 168. AC and BD intersect at G. Given that the area of AGB is 10 more than the area of CGB, find the smallest possible area of hexagon $ABCDEF$.



Solution. Notice that $[ACDF] = [ABDE]$ tells us that $[ABD] = [ACD]$, so B and C are the same distance from AD. Therefore, $AD \parallel BC$. Suppose we let $x = [CGB]$ and $x + 10 = [AGB]$. Then, $[DGA] = \frac{DG}{GB}[AGB] = \frac{AG}{GC}(x + 10) = \frac{(x+10)^2}{x}$. Thus, $[BAD] = [BGA] + [DGA] = (x + 10) + \frac{(x+10)^2}{x} = \frac{1}{2}(168) = 84$. Thus, we have $x^2 - 27x + 50 = 0$, so $(x - 2)(x - 25) = 0$. Thus, $x = 2, 25$. Now, note that $[ABCD] = [ACD] + [ABC] = \frac{1}{2}(168) + x + (x + 10)$, so by letting $x = 2$, this is at least $84 + 2 + 12 = 98$. Now, let O be the midpoint of AD. We note that, because $ACDF$ and $ABDE$ are both parallelograms, O is also the midpoint of BE and CF. This tells us that $ABCD$ and $DEFA$ are symmetric about O, so in particular, they have the same area. Thus, we have $[ABCDEF] = 2[ABCD] = 2(98) = \boxed{196}$. □

§2 Trigonometry

Trigonometry is an extremely useful tool in AIME geometry problems, allowing us to efficiently relate lengths and angles. The definition of trig function using right triangles can be useful from time to time; however, the reason trigonometry is so useful and universal is because of the following two basic facts:

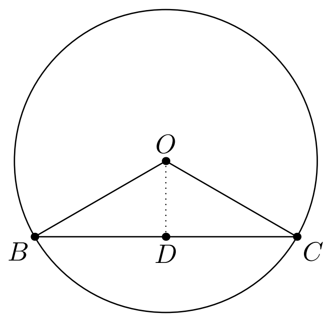
Theorem 2.1 (Extended Law of Sines)

In any triangle ABC, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the circumradius of triangle ABC.

Sketch. We will just show that $a = 2R \sin A$ and the rest follows by symmetry.

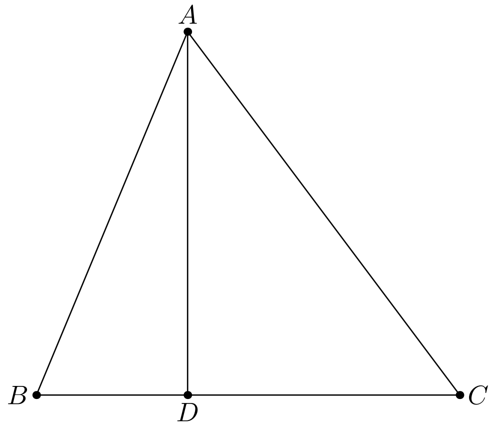


Note that $\angle BOC = 2\angle A$, so the height from O to BC forms an angle of $\angle A$. So, $BC = 2BD = 2R \sin A$, as desired. □

Theorem 2.2 (Law of Cosines)
In any triangle ABC, we have

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Sketch. Draw the height from A to BC



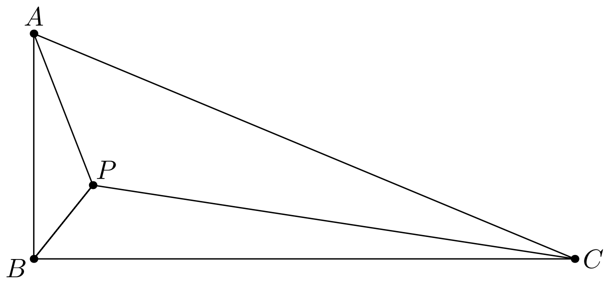
Now, we have $AD = b\sin A$, $CD = b\cos A$, so

$$AB^2 = AD^2 + (a - CD)^2 = b^2\sin^2 A + a^2 - 2ab\cos C + b^2\cos^2 A = a^2 + b^2 - 2ab\cos C$$

as desired. □

These two theorems are enough to solve a wide array of geometric problems that are difficult to approach otherwise.

Example 2.3 (1987 AIME # 9)
Triangle ABC has right angle at B, and contains a point P for which $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.



Solution. Clearly $\angle APB = \angle BPC = \angle CPA = 120°$. Since we know PA and PB, we can now use the Law of Cosines on $\triangle PAB$ to find AB:

$$AB^2 = PA^2 + PB^2 - 2PA \cdot PB\cos\angle APB = 10^2 + 6^2 - 2\cdot 10\cdot 6\left(-\frac{1}{2}\right) = 196$$

and $AB = 14$.

Unfortunately, this is all we can directly compute with the given lengths. However, we still haven't used the fact that $\angle ABC$ is right, or that $\angle APC = \angle BPC = 120°$. How can we bring these facts in? If we set $PC = x$ as a variable, we can write some more equations using these additional facts, and hopefully we'll be able to solve them for x.

Let's use the Law of Cosines on $\triangle BPC$: this gives

$$BC^2 = BP^2 + CP^2 - 2BP \cdot CP \cos \angle BPC = 6^2 + x^2 - 2 \cdot 6 \cdot x \left(-\frac{1}{2}\right) = x^2 + 6x + 36.$$

Similarly, if we use the Law of Cosines on $\triangle APC$ we get

$$AC^2 = AP^2 + CP^2 - 2AP \cdot CP \cos \angle APC = 10^2 + x^2 - 2 \cdot 10 \cdot x \left(-\frac{1}{2}\right) = x^2 + 10x + 100.$$

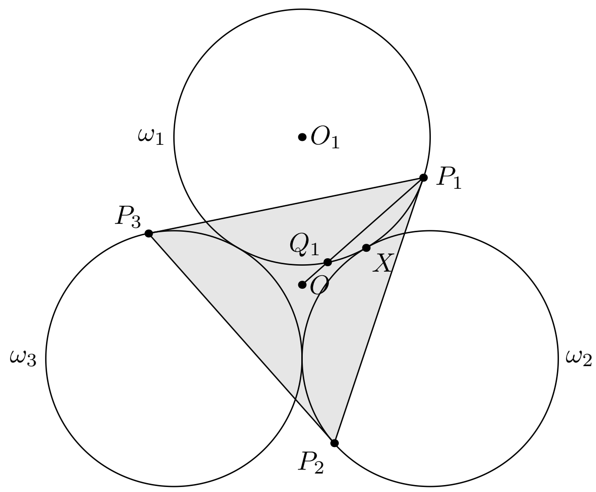
So now we've computed the lengths AB, BC, AC. But we know that $\triangle ABC$ is right, so we can use the Pythagorean Theorem! This allows us to solve for x:

$$196 + (x^2 + 6x + 36) = (x^2 + 10x + 100) \implies 6x + 232 = 10x + 100 \implies x = \boxed{33}.$$

□

Example 2.4 (2018 AMC 12B # 25)

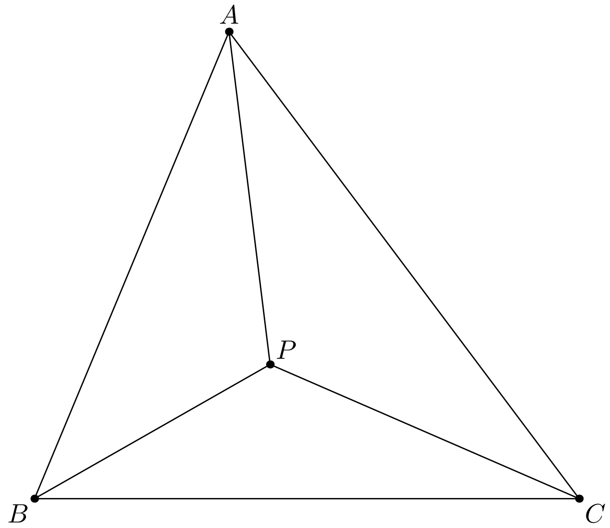
Circles ω_1, ω_2, and ω_3 each have radius 4 and are placed in the plane so that each circle is externally tangent to the other two. Points P_1, P_2, and P_3 lie on ω_1, ω_2, and ω_3 respectively such that $P_1P_2 = P_2P_3 = P_3P_1$ and line P_iP_{i+1} is tangent to ω_i for each $i = 1, 2, 3$, where $P_4 = P_1$. The area of $\triangle P_1P_2P_3$ can be written in the form $\sqrt{a} + \sqrt{b}$ for positive integers a and b. What is $a + b$?



Solution. Suppose the center of the triangle $P_1P_2P_3$ is O, and the center of ω_i is O_i. Then, we have $\angle OP_1P_2 = 30^\circ$. Then, suppose OP_1 intersects ω_1 at some point Q_1 such that $P_1 \neq Q_1$.Then, we have that $\angle OP_1P_2 = \frac{1}{2}\angle Q_1O_1P_1$, so $\angle Q_1O_1P_1 = 60^\circ$. Thus, $P_1Q_1 = 4$. Now, note that the length of the tangent from O to ω_1 is $\frac{4}{\sqrt{3}}$ because we have O_1OX is a 30-60-90 triangle. Thus, the power of O with respect to ω_1 is $\left(\frac{4}{\sqrt{3}}\right)^2 = \frac{16}{3}$. Thus, we also have $OP_1 \cdot OQ_1 = \frac{16}{3}$. Letting $x = OP_1$, we get $x(x-4) = \frac{16}{3}$. Solving this gives $x = 2 + 2\sqrt{\frac{7}{3}}$. Now, we just have to calculate the area of $\triangle P_1P_2P_3$, so we have $[P_1P_2P_3] = 3[P_1OP_2] = 3\left(\frac{1}{2}OP_1 \cdot OP_2 \sin 120^\circ\right) = \frac{3\sqrt{3}}{4}OP_1^2 = 10\sqrt{3} + 6\sqrt{7} = \sqrt{300} + \sqrt{252}$, so our answer is $300 + 252 = \boxed{552}$. □

Example 2.5 (1999 AIME # 14)

Point P is located inside traingle ABC so that angles PAB, PBC, and PCA are all congruent. The sides of the triangle have lengths $AB = 13, BC = 14$, and $CA = 15$, and the tangent of angle PAB is m/n, where m and n are relatively prime positive integers. Find $m + n$.



Solution. Let $\theta = \angle PAB = \angle PBC = \angle PCA$. We want some way to relate a trigonometry function of θ to the lengths of the triangle. We see that we can do that with the Law of Cosines. If we let $PA = x$, $PB = y$, and $PC = z$, then the Law of Cosines on $\triangle PBC$ yields

$$y^2 + 196 - 28y\cos\theta = z^2.$$

Similarly, if we use the law of cosines on $\triangle PAB$ and $\triangle PCA$, we get

$$x^2 + 169 - 26x\cos\theta = y^2,$$

$$z^2 + 225 - 30z\cos\theta = x^2.$$

We will need to get rid of x^2, y^2, and z^2, which we see can be done when we add all three of these equations. We obtain

$$(x^2+y^2+z^2)-590-(26x+28y+30z)\cos\theta = (x^2+y^2+z^2) \implies (26x+28y+30z)\cos\theta = 590.$$

It suffices to determine $26x + 28y + 30z$. We note that the term $26x$ is similar to the area of $\triangle PAB$ because $[PAB] = \frac{1}{2}(13x)\sin\theta$. Now, if we add the areas of $[PAB]$, $[PBC]$, and $[PCA]$, we just get the area of $[ABC]$, so we have

$$[ABC] = [PAB] + [PBC] + [PCA] = \frac{1}{2}(13x + 14y + 15z)\sin\theta = 84.$$

Therefore, $(13x + 14y + 15z)\sin\theta = 168$. Now, if we divide this equation by the other equation we had with $26x + 28y + 30z$. this term cancels out, we and obtain

$$\frac{\tan\theta}{2} = \frac{168}{590} = \frac{84}{295}.$$

Therefore, $\tan\theta = \frac{168}{295}$ and the answer is $\boxed{463}$. □

Remark 2.6. The point described in the previous example is known as a Brocard point, and it is known to exist in every triangle. There is also a second Brocard point Q such that $\angle QBA = \angle QCB = \angle QAC$. Interestingly, the angle for the second Brocard point is the same as for the first Brocard point.

Another equivalent definition for the first Brocard point is to construct the circle passing through A and B that is tangent to BC at B, the circle passing through B and C at that is tangent to CA at C, and the circle passing through C and A that is tangent to AB at A. These circles meet at the first Brocard point.

From the solution of this example, it can be shown that the angle satisfies $\cot\theta = \frac{a^2+b^2+c^2}{4[ABC]}$. Also, we can prove that

$$\cos\theta = \cot A + \cot B + \cot C.$$

§3 Triangle Formulas

Triangles have numerous rich properties and appear frequently in geometry problems. In the AIME, we are often presented with a given triangle and asked to compute various lengths, angles, and areas related to this triangle. The following list of facts and identities, while not comprehensive, can help confidently compute many simple triangular quantities.

In the following formulas, we use the following choice of variables:

- $a = BC,\ b = CA,\ c = AB,\ s = \frac{a+b+c}{2}$
- $A = \angle BAC,\ B = \angle ABC,\ C = \angle ACB$
- R denotes the circumradius, r denotes the inradius

Theorem 3.1 (Area formulas)

In any triangle ABC, if h_a denotes the length of the altitude from A to $\overline{BC}$ and $[ABC]$ denotes the area of $\triangle ABC$ then

$$[ABC] = \frac{1}{2}ah_a = \frac{1}{2}bc\sin A = \sqrt{s(s-a)(s-b)(s-c)} = rs = \frac{abc}{4R}$$

Theorem 3.2 (Ratio Lemma)

If D is any point on line BC then

$$\frac{BD}{CD} = \frac{AB}{AC}\cdot\frac{\sin BAD}{\sin CAD}.$$

Theorem 3.3 (Stewart's Theorem)

Let D be a point on side BC of triangle ABC. If $AD = d,\ BD = m,\ CD = n$ then

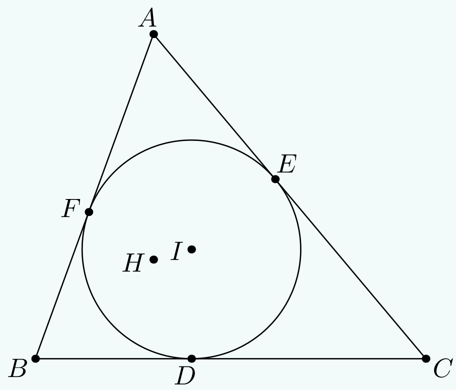
$$b^2m + c^2n = a(d^2 + mn).$$

(This helps us compute the lengths of arbitrary *cevians*, segments with one endpoint at a vertex of the triangle and the other endpoint on the opposite side.)

Theorem 3.4 (some incenter and orthocenter lengths)

Let H and I be the orthocenter and incenter of triangle ABC, respectively. The incircle touches sides BC, CA, AB at D, E, F, respectively. Then:

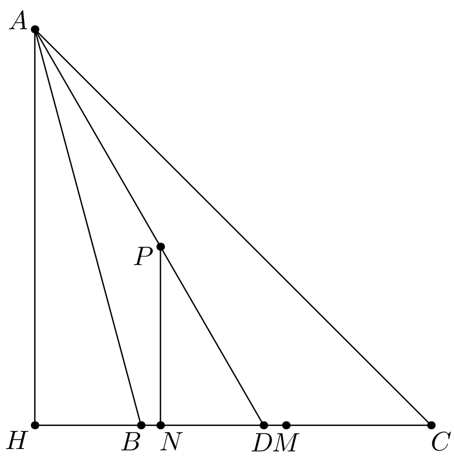
- $AE = AF = s - a$; $BF = BD = s - b$; $CD = CE = s - c$
- $\tan\frac{A}{2} = \frac{r}{s-a}$, $\tan\frac{B}{2} = \frac{r}{s-b}$, $\tan\frac{C}{2} = \frac{r}{s-c}$
- $AH = 2R|\cos A|$, $BH = 2R|\cos B|$, $CH = 2R|\cos C|$



In the following examples, we'll see applications of these triangle identities. As AIME problems, they won't be formula dumps; however, the efficient use of the above formulas will prove essential to obtaining a clean and accurate solution.

Example 3.5 (2014 AIME II # 14)

In $\triangle ABC$, $AB = 10$, $\angle A = 30^\circ$, and $\angle C = 45^\circ$. Let H, D, and M be points on line $\overline{BC}$ such that $\overline{AH} \perp \overline{BC}$, $\angle BAD = \angle CAD$, and $BM = CM$. Point N is the midpoint of segment $\overline{HM}$, and point P is on ray AD such that $\overline{PN} \perp \overline{BC}$. Then $AP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. We're given lots of nice angles: 30°, 45°, 90°, etc. That means many of the triangles in our diagram should be fairly nice. In particular, the two given angle conditions imply $B = 105^\circ$.

Let's first step back and plan how we are going to compute AP. We see that $\overline{AH}$ and $\overline{PN}$ are both perpendicular to $\overline{BC}$ and are thus parallel, so $\triangle ADH \sim \triangle PDN$. But we know that $\angle AHD = 90^\circ$ while

$$\angle ADH = 180^\circ - \angle ABD - \angle BAD = 180^\circ - 105^\circ - \frac{1}{2} \cdot 30^\circ = 60^\circ.$$

This means that $\triangle ADH$ and $\triangle PDN$ are $30^\circ - 60^\circ - 90^\circ$ triangles. In particular,

$$AD = 2HD, \; PD = 2ND \implies AP = AD - PD = 2(HD - ND) = 2HN = HM.$$

So to find AP, we only need to find HM, which is considerably easier because both H and M are nice points on a side of $\triangle ABC$.

It's easier to split HM up into $BH + BM$, as this utilizes the fact that M is the midpoint of $\overline{BC}$. By the Law of Sines,

$$\frac{AB}{\sin C} = \frac{BC}{\sin A} \implies BC = AB \cdot \frac{\sin A}{\sin C} = 10\frac{1/2}{\sqrt{2}/2} = 5\sqrt{2}$$

Thus $BM = \frac{1}{2}BC = \frac{5\sqrt{2}}{2}$. To find BH, we see that $\angle ABH = 180^\circ - B = 75^\circ$ so

$$BH = AB\cos 75^\circ = 10 \cdot \frac{\sqrt{6} - \sqrt{2}}{4} = \frac{5\sqrt{6} - 5\sqrt{2}}{2}.$$

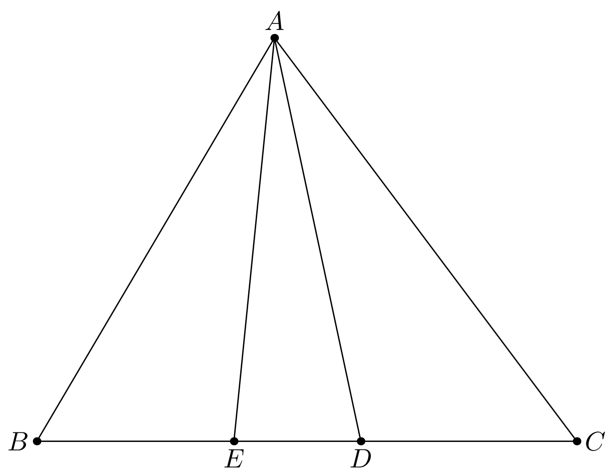
(It's very useful to memorize the trig functions of 15° and 75°, which can be derived from the half-angle formulas). Alternatively, we could've found AC from the Law of Sines and gotten $CH = \frac{\sqrt{2}}{2}AC$ and $\triangle ACH$ is a $45^\circ - 45^\circ - 90^\circ$ triangle, whence $BH = CH - BC$. So now we have

$$AP = HM = BH + BM = \frac{5\sqrt{6} - 5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} = \frac{5\sqrt{6}}{2}$$

so $AP^2 = \frac{75}{2}$ and our answer is $75 + 2 = \boxed{77}$. □

Example 3.6 (2005 AIME II # 14)

In triangle ABC, $AB = 13$, $BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = \frac{p}{q}$ where p and q are relatively prime positive integers, find q.



Solution. We know the lengths AB, BC, CA, BD, CD and also the angle equalities $\angle BAE = \angle CAD$ and $\angle CAE = \angle BAD$. This strongly suggests that the Ratio Lemma will be useful, given that we know so many of the quantities involved in the equation.

Since we want BE, we set up the equation

$$\frac{BE}{CE} = \frac{AB}{AC} \cdot \frac{\sin BAE}{\sin CAE}.$$

However, this is equal to

$$\frac{AB}{AC} \cdot \frac{\sin CAD}{\sin BAD}.$$

We can use the Ratio Lemma again to evaluate the second factor:

$$\frac{AC}{AB} \cdot \frac{\sin CAD}{\sin BAD} = \frac{CD}{BD} \implies \frac{\sin CAD}{\sin BAD} = \frac{CD}{BD} \cdot \frac{AB}{AC} = \frac{6}{9} \cdot \frac{13}{14} = \frac{13}{21}.$$

Thus,

$$\frac{BE}{CE} = \frac{AB}{AC} \cdot \frac{13}{21} = \frac{13}{14} \cdot \frac{13}{21} = \frac{169}{294}.$$

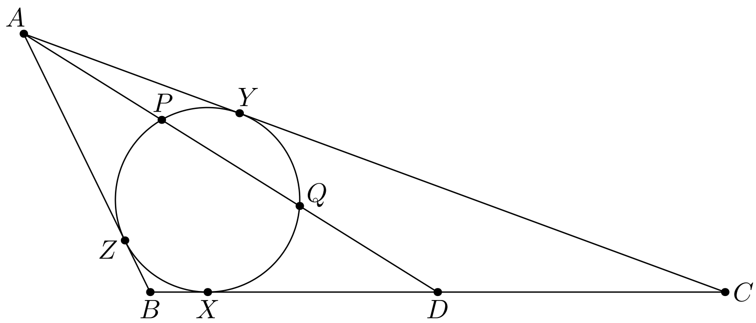
Since $BE + CE = BC$ we now have

$$BE + \frac{294}{169} BE = BC = 15 \implies BE = \frac{2535}{463}$$

and our answer is $\boxed{463}$. □

Example 3.7 (2005 AIME I # 15)

Triangle ABC has $BC = 20$. The incircle of the triangle evenly trisects the median AD. If the area of the triangle is $m\sqrt{n}$ where m and n are integers and n is not divisible by the square of a prime, find $m + n$.



Solution. We may assume that $AB \le AC$ so that the diagram above is accurate (so X is on segment BD rather than segment CD). Let the median intersect the incircle at P and Q, with P closer to A than Q. Then we may define $AP = PQ = QD = x$. When we have a line intersecting a circle, power of a point is a good idea. Suppose the incircle touches $\overline{BC}, \overline{CA}, \overline{AB}$ at X, Y, Z respectively. Using power of a point from A, we get that

$$AY^2 = AZ^2 = AP \cdot AQ = x \cdot 2x \implies AY = AZ = x\sqrt{2}.$$

Similarly, power of a point from D gives

$$DX^2 = DQ \cdot DP = x \cdot 2x \implies DX = x\sqrt{2}.$$

Here, we can make a clever observation that simplifies computation: we've just found that $DX = AY = AZ$. Since $BZ = BX$ and $AZ = DX$, it follows that $AB = BD$. But $BD = \frac{1}{2}BC = 10$, so $AB = 10$ as well.

It remains to find $b = AC$ before we can compute $[ABC]$ using Heron. Here, we'll need to make use of the fact that $AY = x\sqrt{2}$. First, note that $AD = 3x$, and by the median formula we get

$$9x^2 = AD^2 = \frac{2b^2 + 2c^2 - a^2}{4} = \frac{b^2 - 100}{2}.$$

On the other hand, we have

$$x\sqrt{2} = DX = DB - BX = \frac{a}{2} - \frac{a + c - b}{2} = \frac{b - c}{2} = \frac{b - 10}{2}.$$

We now have two expressions involving x and b. We can use them to solve for b:

$$2x^2 = \left(\frac{b - 10}{2}\right)^2, \; 9x^2 = \frac{b^2 - 100}{2}$$

so we get

$$9\left(\frac{b - 10}{2}\right)^2 = b^2 - 36b + 260 = 0.$$

This has the two solutions $b = 10$ and $b = 26$, but we throw out the solution $b = 10$ because it doesn't satisfy the triangle inequality. Now $a = 20, b = 26, c = 10$ and

$$[ABC] = \sqrt{28 \cdot 8 \cdot 2 \cdot 18} = 24\sqrt{14}.$$

So the answer is $24 + 14 = \boxed{38}$. □

§4 Problems

Problem 4.1 (2017 HMMT Geometry # 1)**.** Let A, B, C, D be four points on a circle in that order. Also, $AB = 3$, $BC = 5$, $CD = 6$, and $DA = 4$. Let diagontals AC and BD intersect at P. Compute $\frac{AP}{CP}$.

Problem 4.2 (2018 CMIMC Geometry # 3)**.** Let ABC be a triangle with side lengths $5, 4\sqrt{2}$, and 7. What is the area of the traingle with side lengths $\sin A$, $\sin B$, and $\sin C$?

Problem 4.3 (2013 AIME II # 5)**.** In equilateral $\triangle ABC$ let points D and E trisect $\overline{BC}$. Then $\sin(\angle DAE)$ can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is an integer that is not divisible by the square of any prime. Find $a + b + c$.

Problem 4.4 (2015 AIME II # 11)**.** The circumcircle of acute $\triangle ABC$ has center O. The line passing through point O perpendicular to $\overline{OB}$ intersects lines AB and BC at P and Q, respectively. Also $AB = 5$, $BC = 4$, $BQ = 4.5$, and $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 4.5 (1993 AIME # 15)**.** Let $\overline{CH}$ be an altitude of $\triangle ABC$. Let R and S be the points where the circles inscribed in the triangles ACH and BCH are tangent to $\overline{CH}$. If $AB = 1995$, $AC = 1994$, and $BC = 1993$, then RS can be expressed as m/n, where m and n are relatively prime integers. Find $m + n$

Problem 4.6 (2016 AIME I # 9)**.** Triangle ABC has $AB = 40, AC = 31$, and $\sin A = \frac{1}{5}$. This triangle is inscribed in rectangle $AQRS$ with B on $\overline{QR}$ and C on $\overline{RS}$. Find the maximum possible area of $AQRS$.

Problem 4.7 (2018 HMMT Geometry # 5)**.** In quadrilateral $MARE$ inscribed in a unit circle ω, AM is a diameter of ω, and E lies on the angle bisector of $\angle RAM$. Given that triangles RAM and REM have the same area, find the area of quadrilateral $MARE$.

Problem 4.8 (2018 PUMaC Geometry #4)**.** Triangle ABC has $\angle A = 90^{\circ}$, $\angle C = 30^{\circ}$, and $AC = 12$. Let the circumcircle of this triangle be W. Define D to be the point on arc BC not containing A so that $\angle CAD = 60°$. Define points E and F to be the foots of the perpendiculars from D to lines AB and AC, respectively. Let J be the intersection of line EF with W, where J is on the minor arc AC. The line DF intersects W at H other than D. The area of the triangle FHJ is in the form $\frac{a}{b}(\sqrt{c} - \sqrt{d})$ for positive integers a,b,c,d, where a, b are relatively prime, and the sum of a, b, c, d is minimal. Find $a + b + c + d$.

Problem 4.9 (2017 CMIMC Geometry #5)**.** Tow circles ω_1 and ω_2 are said to be *orthogonal* if they intersect each other at right angles. In other words, for any point P lying on both ω_1 and ω_2, if ℓ_1 is the line tangent to ω_1 at P and ℓ_2 is the line tangent to ω_2 at P, then $\ell_1 \perp \ell_2$. (Two circles which do not intersect are not orthogonal)
Let $\triangle ABC$ be a triangle with area 20. Orthogonal circles ω_B and ω_C are drawn with ω_B centered at B and ω_C centered at C. Poitns T_B and T_C are placed on ω_B and ω_C respectively such that AT_B is tangent to ω_B and AT_C is tangent to ω_C. If $AT_B = 7$ and $AT_C = 11$, what is $\tan BAC$?

Problem 4.10 (2018 PUMaC Geometry #5)**.** Let $\triangle ABC$ be a triangle with side lengths $AB = 9$, $BC = 10$, $CA = 11$. Let O be the circumcenter of $\triangle ABC$. Denote $D = AO \cap BC$, $E = BO \cap CA$, $F = CO \cap AB$. If $\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF}$ can be written in simplest form as $\frac{a\sqrt{b}}{c}$, find $a + b + c$.

Problem 4.11 (2017 HMMT Geometry # 4)**.** Let $ABCD$ be a convex quadrilateral with $AB = 5, BC = 6, CD = 7$, and $DA = 8$. Let M, P, N, Q be the midpoints of sides AB, BC, CD, DA respectively. Compute $MN^2 - PQ^2$.

Problem 4.12 (2017 PUMaC Geometry #5)**.** Rectangle $HOMF$ has $HO = 11$ and $OM = 5$. Triangle ABC has orthocenter H and circumcenter O. M is the midpoint of BC and altitude AF meets BC at F. Find the length of BC.

Problem 4.13 (2016 PUMaC Geometry # 7)**.** Let $ABCD$ be a cyclic quadrilateral with circumcircle ω and let AC and BD intersect at X. Let the line through A parallel to BD intersect line CD at E and ω at $Y \neq A$. If $AB = 10$, $AD = 24$, $XA = 17$, and $XB = 21$, then the area of $\triangle DEY$ can be writtein in simplest form as $\frac{m}{n}$. Find $m + n$.

Problem 4.14 (2017 HMMT Geometry # 8)**.** Let ABC be a triangle with circumradius $R = 17$ and inradius $r = 7$. Find the maximum possible value of $\sin \frac{A}{2}$.

Problem 4.15 (2015 AIME II # 15)**.** Circles $\mathcal{P}$ and $\mathcal{Q}$ have radii 1 and 4, respectively, and are externally tangent at point A. Point B is on $\mathcal{P}$ and point C is on $\mathcal{Q}$ so that line BC is a common external tangent of the two circles. A line ℓ through A intersects $\mathcal{P}$ again at D and intersects $\mathcal{Q}$ again at E. Points B and C lie on the same side of ℓ, and the areas of $\triangle DBA$ and $\triangle ACE$ are equal. This common area is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 4.16 (IMO 1961/2)**.** Let a, b, c be the sides of a triangle, and S its area. Prove:

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3}.$$

In what case does equality hold?