Equations B

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Last week, the equations we focused on were many polynomial or rational in nature: they only involved the four basic arithmetic operations of addition, subtraction, multiplication, and division. In this handout, we discuss equations that involve other types of operations.

§1 Radicals

The general strategy in this section is to get rid of radicals.

Example 1.1 (2015 MPFG #17)
Let S be the sum of all distinct real solutions of the equation

$$\sqrt{x+2015} = x^2 - 2015.$$

Compute $\lfloor 1/S \rfloor$.

Solution. We begin by squaring both sides to remove radicals, giving

$$x + 2015 = (x^2 - 2015)^2$$

Now, if we suppose $f(x) = x^2 - 2015$, we see that the above equation may be rearranged into

$$f(f(x)) = x$$

Now, note that if $f(x) = x$, this equation is satisfied. Thus, we note that $f(f(x)) - x$ must be factorizable into $(f(x) - x)g(x)$ for some quadratic g. Now, plugging back in $f(x) = x^2 - 2015$ gives

$$f(f(x)) - x = (x^2 - x - 2015)(x^2 + x - 2014)$$

Now, note that we only care about roots for which $x^2 - 2015 \geq 0$. Thus, we consider our roots

$$x = \frac{1 \pm \sqrt{8061}}{2}, \frac{-1 \pm \sqrt{8057}}{2}$$

and we find that only

$$x = \frac{1 + \sqrt{8061}}{2}, \frac{-1 - \sqrt{8057}}{2}$$

work. Thus,

$$S = \frac{\sqrt{8061} - \sqrt{8057}}{2}$$

so

$$\frac{1}{S} = \frac{2}{\sqrt{8061} - \sqrt{8057}} = \frac{\sqrt{8061} + \sqrt{8057}}{2}$$

Thus, $\left\lfloor \frac{1}{S} \right\rfloor = 89$. □

§2 Floors

Example 2.1 (2019 HMMT November Team #5)

Compute the sum of all positive real numbers $x \le 5$ satisfying

$$x = \frac{\lceil x^2 \rceil + \lceil x \rceil \cdot \lfloor x \rfloor}{\lceil x \rceil + \lfloor x \rfloor}.$$

Solution. We begin by noting that all $x = 1, 2, 3, 4, 5$ will work. For all other x, we have that $\lceil x \rceil = \lfloor x \rfloor + 1$, so we do casework on what they are.

- $\lfloor x \rfloor = 0, \lceil x \rceil = 1$
  Note that in this case, we must have $0 < x < 1$, so $0 < x^2 < 1$ and $\lceil x^2 \rceil = 1$. Thus, we find that $x = \frac{1+1\cdot 0}{0+1} = 1$, which doesn't satisfy $\lfloor x \rfloor = 0$, so we have no x in this case.

- $\lfloor x \rfloor = 1, \lceil x \rceil = 2$
  Note that in this case, $1 < x^2 < 4$, so $\lceil x^2 \rceil = 2, 3, 4$. Plugging in each of these gives $x = \frac{4}{3}, \frac{5}{3}, \frac{6}{3}$, respectively. We find that $x = \frac{4}{3}, \frac{5}{3}$ both work, but $x = \frac{6}{3} = 2$ does not in this case, since we need $\lfloor x \rfloor = 1$.

- $\lfloor x \rfloor = 2, \lceil x \rceil = 3$
  Note that the possible values of $\lceil x^2 \rceil$ are $5, 6, 7, 8, 9$. Plugging each of these in gives $x = \frac{11}{5}, \frac{12}{5}, \frac{13}{5}, \frac{14}{5}, \frac{15}{5}$. Note that, as in the other cases, $\frac{15}{5}$ doesn't work because we need $\lfloor x \rfloor = 2$, but all the other possible x can be verified to work.
  We begin to notice a pattern, in particular, that all possibilities should work, except for when x ends up being an integer.

Thus, we make the following conjecture:

Conjecture. $x = n + \frac{k}{2n+1}$ works for all integer $n > 0$ and $0 \le k \le 2n$.

Proof. We note that $x^2 = n^2 + \frac{2nk}{2n+1} + \frac{k^2}{(2n+1)^2}$. Note that $\frac{2nk}{2n+1} > k - 1$ (simple to show by cross multiplication) and $k - \frac{2nk}{2n+1} = \frac{k}{2n+1} \ge \frac{k^2}{(2n+1)^2}$. Thus, $\lceil x^2 \rceil = n^2 + k$. Now, plugging in $\lfloor x \rfloor = n$ and $\lceil x \rceil = n + 1$ satisfies the given equation. □

Thus, we find the sum of our desired x in the interval $[n, n+1)$ is

$$\sum_{k=0}^{2n} \left(n + \frac{k}{2n+1} \right) = (2n+1)(n) + \frac{1}{2n+1} \sum_{k=0}^{2n} k = 2n^2 + 2n = 2n(n+1),$$

so our answer is

$$\sum_{n=1}^{4} 2n(n+1) + 5 = 85.$$

□

Example 2.2 (2012 HMMT Algebra #5)

Find all ordered triplets (a, b, c) of positive reals that satisfy: $\lfloor a \rfloor bc = 3$, $a\lfloor b \rfloor c = 4$, $ab\lfloor c \rfloor = 5$.

Solution. Another sometimes helpful observation is that $\lfloor x \rfloor$ is about x, and slightly smaller. If we multiply all three equations, then we can approximate $abc:$

$$60 = a^2b^2c^2\lfloor a \rfloor\lfloor b \rfloor\lfloor c \rfloor \geq (\lfloor a \rfloor\lfloor b \rfloor\lfloor c \rfloor)^3 \implies \lfloor a \rfloor\lfloor b \rfloor\lfloor c \rfloor \leq 3.$$

Since all three floors are integers, we now know that at least two of $\lfloor a \rfloor, \lfloor b \rfloor, \lfloor c \rfloor$ are 1 (otherwise $\lfloor a \rfloor\lfloor b \rfloor\lfloor c \rfloor$ would be at least 4). So we can consider cases on which two floors are 1. Before we jump into the cases, a quick review: if we know ab, bc, ca then we can find a, b, c as follows. We multiply ab, bc, ca to find $a^2b^2c^2$, then take the square root to find the value of abc. We can then divide this by ab, bc, ca to find a, b, c.

If $\lfloor a \rfloor = \lfloor b \rfloor = 1$, then $bc = 3$ and $ac = 4$. Now $\lfloor c \rfloor$ must equal $1, 2,$ or 3, so the third equation gives that ab must equal $5, \frac{5}{2}$, or $\frac{5}{3}$, respectively. Since $\lfloor a \rfloor = \lfloor b \rfloor = 1$ we have $ab < 2 \cdot 2 = 4$ so $ab = 5$ is impossible. If $ab = \frac{5}{2}$ then

$$abc = \sqrt{(ab)(bc)(ca)} = \sqrt{\frac{5}{2} \cdot 3 \cdot 4} = \sqrt{30}$$

so

$$a = \frac{abc}{bc} = \frac{\sqrt{30}}{3}, \ b = \frac{abc}{ac} = \frac{\sqrt{30}}{4}, \ c = \frac{abc}{ab} = \frac{2\sqrt{30}}{5}.$$

We can verify that these indeed satisfy $\lfloor a \rfloor = 1$, $\lfloor b \rfloor = 1$, and $\lfloor c \rfloor = 2$ (this is important!), so this is a solution. On the other hand, if $ab = \frac{5}{3}$ then we find that

$$a = \frac{2\sqrt{5}}{3}, \ b = \frac{\sqrt{5}}{2}, \ c = \frac{6\sqrt{5}}{5},$$

but this is *not* a solution because $\lfloor c \rfloor = 2$ instead of $\lfloor c \rfloor = 3$. Thus $(a, b, c) = \left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{4}, \frac{2\sqrt{30}}{5}\right)$ is the only solution in this case.
Our other cases are very similar: we find that there are a couple of possible values for ab, bc, ca, solve for a, b, c, and then check if they satify the floor conditions.

If $\lfloor b \rfloor = \lfloor c \rfloor = 1$ then $ac = 4$ and $ab = 5$. Since $\lfloor a \rfloor$ is $1, 2,$ or 3 we see that bc is $3, \frac{3}{2}$, or 1, respectively. If $\lfloor a \rfloor = 1$ and $bc = 3$ then we get

$$a = \frac{2\sqrt{15}}{3}, \ b = \frac{\sqrt{15}}{2}, \ c = \frac{2\sqrt{15}}{5}$$

but this is not a solution because $\lfloor a \rfloor \neq 1$. If $\lfloor a \rfloor = 2$ and $bc = \frac{3}{2}$ then

$$a = \frac{2\sqrt{30}}{3}, \ b = \frac{\sqrt{30}}{4}, \ c = \frac{\sqrt{30}}{5}$$

but this doesn't work because $\lfloor a \rfloor \neq 2$. Finally, if $\lfloor a \rfloor = 3$ then $bc = 1$ and we get

$$a = 2\sqrt{5}, \ b = \frac{\sqrt{5}}{2}, \ c = \frac{2\sqrt{5}}{5}$$

which fails because $\lfloor a \rfloor \neq 3$. Thus there are no solutions in this case.

Our final case is if $\lfloor c \rfloor = \lfloor a \rfloor = 1$, where $ab = 5$ and $bc = 3$. Here we have that $\lfloor b \rfloor$ must be $1, 2$, or 3, corresponding to $ac = 4, 2, \frac{4}{3}$, respectively. The first case is impossible, like in the first case: We have $ac < 2 \cdot 2 = 4$. For the second case, we solve

$$a = \frac{\sqrt{30}}{3},\ b = \frac{\sqrt{30}}{3},\ c = \frac{\sqrt{30}}{5}$$

which works: we verify $\lfloor a \rfloor = \lfloor c \rfloor = 1$ and $\lfloor b \rfloor = 2$. On the other hand, if $ac = \frac{4}{3}$ then

$$a = \frac{2\sqrt{5}}{3},\ b = \frac{3\sqrt{5}}{2},\ c = \frac{2\sqrt{5}}{5}$$

but this doesn't work because $\lfloor c \rfloor = 0$.

To conclude, we have the two solutions

$$\left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{4}, \frac{\sqrt{30}}{5}\right) \text{ and } \left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{2}, \frac{\sqrt{30}}{5}\right).$$

Essentially, we multiplied the three equations at the beginning to place relatively tight bounds on $\lfloor a \rfloor$, $\lfloor b \rfloor$, and $\lfloor c \rfloor$ and then used casework on these possibilities to find the solutions. □

Example 2.3 (2012 AIME II #10)

Find the number of positive integers n less than 1000 for which there exists a positive real number x such that $n = x\lfloor x \rfloor$.

Solution. Again, the fact that $\lfloor x \rfloor$ is always an integer is useful, specifically taking cases on $\lfloor x \rfloor$. We can rewrite this as

$$x\lfloor x \rfloor = \begin{cases} 0 & 0 \le x < 1 \\ x & 1 \le x < 2 \\ 2x & 2 \le x < 3 \\ 3x & 3 \le x < 4 \\ \dots \end{cases}$$

Now taking cases on $\lfloor x \rfloor$, we can easily see what (real!) values $x\lfloor x \rfloor$ can take:

- $\lfloor x \rfloor = 0 : x\lfloor x \rfloor = 0$.
- $\lfloor x \rfloor = 1 : x\lfloor x \rfloor = x$ and $1 \le x < 2$ so $1 \le x\lfloor x \rfloor < 2$.
- $\lfloor x \rfloor = 2 : x\lfloor x \rfloor = 2x$ and $2 \le x < 3$ so $4 \le x\lfloor x \rfloor < 6$.
- $\lfloor x \rfloor = 3 : x\lfloor x \rfloor = 3x$ and $3 \le x < 4$ so $9 \le x\lfloor x \rfloor < 12$.
- $\dots$

Thus, we see that the positive integers $x\lfloor x \rfloor$ can take are $n^2, n^2+1, \dots, n^2+n-1$ for each positive integer n. Thus there are n integers of the form $x\lfloor x \rfloor$ in each interval $[n^2, (n+1)^2)$. Summing $n = 1, 2, \dots, 30$, we see that there are $1 + 2 + \cdots + 30 = 465$ such integers between 1 and 960. For $n = 31$, we see that the 31 integers $961, 962, \dots, 991$ are also all expressible, and there are no other expressible integers from 992 to 1000, for a total of $465 + 31 = 496$. □

Example 2.4 (Adithya Balachandran)
Let a be a real number such that

$$\lfloor a \rfloor^4 - 4a^2 + \lfloor a \rfloor - 1 = 0.$$

Then, the sum of the squares of all such real numbers a can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the value of $m + n$.

Solution. We know that $\lfloor a \rfloor$ is an integer, so this motivates us to try to isolate all of the terms containing $\lfloor a \rfloor$ on one side.

$$4a^2 = \lfloor a \rfloor^4 + \lfloor a \rfloor - 1$$

$$a^2 = \frac{\lfloor a \rfloor^4 + \lfloor a \rfloor - 1}{4}$$

$$a = \pm\frac{\sqrt{\lfloor a \rfloor^4 + \lfloor a \rfloor - 1}}{2}$$

The reason that this relation is useful is that when a is large, the RHS is approximately $\lfloor a \rfloor^2$, which is much larger than the LHS. Therefore, we expect that we can perform some sort of bounding on the value of $\lfloor a \rfloor$.

Let's assume that we are first taking the positive root in the above equation. First, note that $0 \le a - \lfloor a \rfloor < 1$, so we have that

$$0 \le \frac{\sqrt{\lfloor a \rfloor^4 + \lfloor a \rfloor - 1}}{2} - \lfloor a \rfloor < 1.$$

For positive values of a we see that $\lfloor a \rfloor = 2$ is the smallest value that works, and for $\lfloor a \rfloor \ge 3$, the fourth power terms grows faster, so the inequality will not be satisfied. Therefore, no other value works. Now, lets take the negative value for the square root.

$$0 \le -\frac{\sqrt{\lfloor a \rfloor^4 + \lfloor a \rfloor - 1}}{2} - \lfloor a \rfloor < 1$$

Now, for $\lfloor a \rfloor > 0$, the equation is negative, so none of those values work. We now look into negative integers. We see that $\lfloor a \rfloor = -2$ works, and as $\lfloor a \rfloor$ decreases, the fourth power term increases faster, so the inequality can never be achieved again. Therefore, we have that the solutions are $\lfloor a \rfloor = \pm 2$. Substituting this into the equation $a = \pm\frac{\sqrt{\lfloor a \rfloor^4 + \lfloor a \rfloor - 1}}{2}$, we have that

$$a = -\frac{\sqrt{13}}{2}, \frac{\sqrt{17}}{2}$$

The sum of the squares of the solutions is $\frac{13}{4} + \frac{17}{4} = \frac{30}{4} = \frac{15}{2}$, so the answer to the problem is 17. □

§3 Exponents and Logarithms

We will first discuss some exponent and logarithm properties that you need to be familiar with in order to solve problems on this topic.

1. Exponent addition when multiplying two numbers with the same base: $a^x \cdot a^y = a^{x+y}$
2. Exponent subtraction when dividing two numbers with the same base: $\frac{a^x}{a^y} = a^{x-y}$.

3. Multiplication with the same exponent: $a^x \cdot b^x = (ab)^x$.
4. Logarithm addition: $\log_b x + \log_b y = \log_b(xy)$.
5. Logarithm subtraction: $\log_b x - \log_b y = \log_b \left(\frac{x}{y}\right)$.
6. Change of base: $\frac{\log_c a}{\log_c b} = \log_b a$.
7. Logarithm power rules: $\log_b x^a = a \log_b x$ and $\log_{b^a} x = \frac{1}{a} \log_b x$.
8. Reciprocal of a logarithm: $\frac{1}{\log_b a} = \log_a b$.

We'll also mention another useful tip for switching back and forth between logarithms and exponents. Logarithms and exponents are inverse functions. That is, if $\log_a b = c$, then $a^c = b$, which can also be written as $a^{\log_a b} = b$. We will leave it as an exercise to the reader to prove all of these properties.

Example 3.1 (2002 AIME I #6)
The solutions to the system of equations

$$\log_{225} x + \log_{64} y = 4$$
$$\log_x 225 - \log_y 64 = 1$$

are (x_1, y_1) and (x_2, y_2). Find $\log_{30}(x_1 y_1 x_2 y_2)$.

Solution. First, we note that we can use the logarithm power rules shown above to reduce the equations to the following:

$$\frac{1}{2}\log_{15} x + \frac{1}{6}\log_2 y = 4,$$
$$2\log_x 15 - 6\log_y 2 = 1.$$

In order to avoid rewriting the logarithms over and over again, we will let $a = \log_{15} x$ and $b = \log_2 y$. Then, the first equation we are given is $\frac{1}{2}a + \frac{1}{6}b = 4$. By the reciprocal of a logarithm property,

$$\log_x 15 = \frac{1}{a}, \qquad \log_y 2 = \frac{1}{b}.$$

Therefore, the second equation we are given is $\frac{2}{a} - \frac{6}{b} = 1$. We can multiply this equation by ab to obtain $2b - 6a = ab$. We now have two equations for a and b. We can use the equation $\frac{1}{2}a + \frac{1}{6}b = 4$ to solve for b and substitute this in to the second equation. We get$b = 24 - 3a$. Therefore,

$$2(24 - 3a) - 6a = a(24 - 3a) \implies 3a^2 - 36 + 48 = 0.$$

Now, we can divide by 3 to obtain $a^2 - 12a + 24 = 0$. Note that the roots of this polynomial are not integers, so we will not solve for the roots directly. The roots of this polynomial are $\log_{15} x_1$ and $\log_{15} x_2$. We want to determine the value of the product $x_1 x_2$. By the logarithm addition property and Vieta's,

$$\log_{15}(x_1 x_2) = \log_{15} x_1 + \log_{15} x_2 = 12 \implies x_1 x_2 = 15^{12}.$$

Now, we can also determine the product y_1y_2. As we had the substitution $b = 24 - 3a$ above,

$$\begin{aligned}\log_2(y_1y_2) = \log_2 y_1 + \log_2 y_2 &= 24 - 3\log_{15} x_1 + 24 - 3\log_{15} x_2 \\ &= 48 - 3(\log_{15} x_1 + \log_{15} x_2) \\ &= 12.\end{aligned}$$

Therefore, $y_1y_2 = 2^{12}$. Now, this means

$$\log_{30}(x_1y_1x_2y_2) = \log_{30}(2^{12}5^{12}) = \log_{30}(30^{12}) = 12.$$

□

Example 3.2 (2016 PUMaC Algebra #3)

For positive real numbers x and y, let $f(x, y) = x^{\log_2 y}$. Compute the sum of the solutions to the equation

$$4096f(f(x, x), x) = x^{13}$$

Solution. The function $x^{\log_2 y}$ may seem unusual, but this might remind you of the property $a^{\log_a b} = b$. However, the problem here is that the base of the exponent is x instead of 2. Therefore, this motivates us to try to make the substitution $x = 2^a$, so that the base becomes 2. Therefore, we have the following nice simplification for $f(x, y)$:

$$f(x, y) = f(2^a, y) = 2^{a\log_2 y} = 2^{\log_2 y^a} = y^a$$

Therefore, $f(x, x) = f(2^a, x) = x^a = 2^{a^2}$ by substituting $y = x$ in the relation above for $f(x, y)$. From the same relation, we also ahve

$$f(f(x, x), x) = f(2^{a^2}, x) = x^{a^2} = 2^{a^3}.$$

Noting that $4096 = 2^{12}$,

$$4096f(f(x, x), x) = 2^{12}2^{a^3} = 2^{a^3+12} = x^{13} = 2^{13a}$$

The function $y = 2^x$ is strictly increasing, so the exponents must be equal. This means that $a^3 + 12 = 13a$. We can factor this cube as $(a - 1)(a - 3)(a + 4)$. Therefore, $a = -4, 1, 3$. The values for x are $2^{-4} = \frac{1}{16}$, $2^1 = 2$, and $2^3 = 8$. The sum of these solutions is $\frac{1}{16} + 2 + 8 = \frac{161}{16}$. □

Example 3.3 (2017 AIME I #14)

Let $a > 1$ and $x > 1$ satisfy $\log_a(\log_a(\log_a 2) + \log_a 24 - 128) = 128$ and $\log_a(\log_a x) = 256$. Find the remainder when x is divided by 1000.

Solution. In this solution, we will demonstrate the technique of repeated substitution and use logarithm properties. First, note that we can actually clear the outer logarithm by using the fact that logarithms and exponents are inverse functions.

$$\log_a(\log_a 2) + \log_a(24) - 128 = a^{128}$$

Motivated by the substitution in the previous problem, we will let $a = 2^b$. This is because this substitution will allow us to use many logarithm properties to simplify the equation. The equation simplifies to

$$\begin{aligned} 2^{128b} &= \log_{2^b}(\log_{2^b} 2) + \log_{2^b}(24) - 128 \\ &= \log_{2^b}\left(\frac{1}{b}\right) + \log_{2^b}(24) - 128 \\ &= \log_{2^b}\left(\frac{24}{b}\right) - 128 \\ &= \log_{2^b}\left(\frac{24}{b}\right) - \log_{2^b}(2^{128b}) \\ &= \log_{2^b}\left(\frac{24}{b2^{128b}}\right) \end{aligned}$$

Now, if we again use the inverse property of logarithms and exponents,

$$(2^b)^{2^{128b}} = 2^{b2^{128b}} = \frac{24}{b2^{128b}}$$

Now, we see the term $b2^{128b}$ on both sides, so we will make the substitution $c = b2^{128b}$. Therefore,

$$2^c = \frac{24}{c} \implies c2^c = 24.$$

Now, if we look at this equation, we see by inspection that the only solution is $c = 3$ because both c and 2^c are monotonically increasing functions. Therefore, $c = 3$ and $b2^{128b} = 3$. In order for this equation to work, we must have $\frac{3}{b}$ be a power of 2. Therefore, let $b = 3 \cdot 2^k$. Therefore,

$$3 \cdot 2^k 2^{128 \cdot 3 \cdot 2^k} = 3 \implies 2^{k + 384 \cdot 2^k} = 1$$

We must have $k + 384 \cdot 2^k = 0$. By inspection, we see that $k = -6$ is the solution to this equation. We notice this because 384 only has 7 powers of 2 so it suffices to try values of k from 0 to 7. Therefore, $b = 3 \cdot 2^{-6} = \frac{3}{64}$, which implies $a = 2^{\frac{3}{64}}$.

Now, we can look at the equation $\log_a(\log_a x) = 256$ to determine x. We can use the inverse property to obtain $a^{256} = \log_a x$. If we use the same property once again we obtain

$$x = a^{a^{256}} = a^{2^{256 \cdot \frac{3}{64}}} = a^{\cdot 2^{12}} = 2^{\frac{3}{64} \cdot 2^{12}} = 2^{192}.$$

To find this mod 1000, it suffices to find this mod 8 and mod 125. 2^{192} is trivially 0 (mod 8). For mod 125, we can utilize Euler's theorem ($2^{100} \equiv 1 \pmod{125}$), so

$$2^{192} \equiv \frac{1}{2^8} = \frac{1}{256} = \frac{1}{6} \equiv 21 \pmod{125}.$$

By the Chinese Remainder Theorem, we obtain $x \equiv 876 \pmod{1000}$.

□

§4 Problems

Problem 4.1 (2020 AMC 12A #13)**.** There are integers a, b, and c, each greater than 1, such that

$$\sqrt[a]{N\sqrt[b]{N\sqrt[c]{N}}} = \sqrt[36]{N^{25}}$$

for all $N > 1$. What is b?

Problem 4.2 (2005 AIME I #8)**.** The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Given that their sum is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

Problem 4.3 (2014 HMMT Algebra #3)**.** Let

$$A = \frac{1}{6}((\log_2(3))^3 - (\log_2(6))^3 - (\log_2(12))^3 + (\log_2(24))^3).$$

Compute 2^A.

Problem 4.4 (2002 AIME II #8)**.** Find the least positive integer k for which the equation $\lfloor \frac{2002}{n} \rfloor = k$ has no integer solutions for n. (The notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x.)

Problem 4.5 (2020 AMC 12A #25)**.** The number $a = \frac{p}{q}$, where p and q are relatively prime positive integers, has the property that the sum of all real numbers x satisfying

$$\lfloor x \rfloor \cdot \{x\} = a \cdot x^2$$

is 420, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. What is $p + q$?

Problem 4.6 (2017 CMIMC Algebra #5)**.** The set S of positive real numbers x such that

$$\left\lfloor \frac{2x}{5} \right\rfloor + \left\lfloor \frac{3x}{5} \right\rfloor + 1 = \lfloor x \rfloor$$

can be written as $S = \bigcup_{j=1}^{\infty} I_j$, where the I_i are disjoint intervals of the form $[a_i, b_i) = \{x \mid a_i \le x < b_i\}$ and $b_i \le a_{i+1}$ for all $i \ge 1$. Find $\sum_{i=1}^{2017}(b_i - a_i)$.

Problem 4.7 (2012 Baltic Way/3)**.** Show that the equation

$$\lfloor x \rfloor (x^2 + 1) = x^3,$$

where $\lfloor x \rfloor$ denotes the largest integer not larger than x, has exactly one real solution in each interval between consecutive positive integers.

Problem 4.8 (2013 USAMTS Round 1 #4)**.** Show that for all integers k and $m \ge 3$, there exists exactly one pair of positive integers (n, i) with $i \le m$ such that

$$\left\lfloor \frac{2^m - 1}{2^{i-1}} n - 2^{m-i} + 1 \right\rfloor = k$$

Problem 4.9 (Canada 1999/1)**.** Find all real solutions to the equation $4x^2 - 40\lfloor x \rfloor + 51 = 0$.

Problem 4.10 (Saint Petersburg 2013/11/1)**.** Find the minimum positive noninteger root of $\sin x = \sin\lfloor x \rfloor$.

Problem 4.11 (2006 MOP)**.** Find all pairs (a, b) of positive real numbers such that $\lfloor a\lfloor bn\rfloor\rfloor = n - 1$ for all positive integers n.

Problem 4.12 (Pan Africa 2012/3)**.** Find all real solutions x to the equation $\lfloor x^2 - 2x\rfloor + 2\lfloor x\rfloor = \lfloor x\rfloor^2$.

Problem 4.13 (Vietnam 2007/1)**.** Solve the system of equations:

$$\begin{cases} \left(1 + \frac{12}{3x+y}\right)\sqrt{x} = 2 \\ \left(1 - \frac{12}{3x+y}\right)\sqrt{y} = 6 \end{cases}$$

Problem 4.14 (Romania TST 2012/1)**.** Prove that for any positive integer $n \geq 2$ we have that

$$\sum_{k=2}^{n} \lfloor \sqrt[k]{n} \rfloor = \sum_{k=2}^{n} \lfloor \log_k n \rfloor.$$