Equations

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§1 Algebraic Manipulations

There are rarely systematic ways to solve general equations or systems of equations, like those exhibited with linear equations or quadratics. However, many contest equations are contrived in a way such that they are made easy with careful manipulations. Thus, being familiar with some of the more common algebraic tricks can go a long way to solving general classes of equations.

Due to this unit's open ended nature, we don't really have many rigorous theorems. However, there are a number of important ideas to keep in mind when solving algebraic equations:

- If the condition is symmetric, try to keep it symmetric. Sometimes, making a substitution to create symmetry is also important. (*e.g.* Given a quadratic $(x-5)(x-1) = 7$ we can "create symmetry" by substituting $y = x - 3$ to center the LHS around 0.)
- Often times, an equations problem will not ask you to solve for each variable, rather an expression containing them, such as their sum. In this case, it is rarely necessary to actually solve for each individual variable. Use the answer form to your advantage when solving, and try to transform what you are given to what you need
- Remember your factorizations so you can spot factorable expressions when working with big algebraic expressions. (*e.g.* The expression $a^3 + b^3 + c^3 - 3abc$ looks a lot more tenable when we realize it is $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$)
- Know common substitutions associated with certain conditions. For example, when given $xyz = 1$, a common substitution would be to set $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$. Solving for a, b, c (up to scaling) can then get us x, y, z.
  There exist many other useful, though less common substitutions, such as $xyz = x+y+z+2 \implies \exists\ a, b, c$ such that $x = \frac{b+c}{a}$, $y = \frac{c+a}{b}$, $z = \frac{a+b}{c}$. It's not necessarily worth memorizing these, but just be on the lookout for substitutions like these that transform messy conditions into easier ones (or eliminate constraints like $xyz = 1$ entirely).

§2 Polynomials

We begin with the simplest type of equations: polynomials. While there are several theorem regarding the roots of polynomials (such as the Rational Root Theorem or Descartes' Rule of Signs), we'll focus on techniques that apply to a broader range of problems: substitution, factoring, and the like.

Example 2.1

Solve $64x^6 + 64x^5 - 320x^4 - 448x^3 - 80x^2 + 4x + 1 = 0$.

Solution. Note that if we substitute $u = 2x$, we will decrease all the coefficients while still keeping them integers. We get the new equation is

$$u^6 + 2u^5 - 20u^4 - 56u^3 - 20u^2 + 2u + 1 = 0$$

Now, note that this sequence of coefficients is symmetric, so we center the degrees of the coefficients about 0, so we divide by u^3. Now, we get

$$\left(u^3 + \frac{1}{u^3}\right) + 2\left(u^2 + \frac{1}{u^2}\right) - 20\left(u + \frac{1}{u}\right) - 56 = 0$$

Now, note that each of these terms may be expressed in terms of $v = u + \frac{1}{u}$. Thus, we have

$$(v^3 - 3v) + 2(v^2 - 2) - 20v - 56 = 0$$

so $v^3 + 2v^2 - 23v - 60 = 0$. Now, we test possible values of v until we see that

$$(v + 3)(v + 4)(v - 5) = 0$$

Thus, we have $u + \frac{1}{u} = -3, -4, 5$, so $u = \frac{-3\pm\sqrt{5}}{2}, -2 \pm \sqrt{3}, \frac{5\pm\sqrt{21}}{2}$, so

$$x = \frac{-3 \pm \sqrt{5}}{4}, \frac{-2 \pm \sqrt{3}}{2}, \frac{5 \pm \sqrt{21}}{4}.$$

□

Example 2.2 (2000 AIME II # 13)

The equation $2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$ has exactly two real roots, one of which is $\frac{m+\sqrt{n}}{r}$, where m, n and r are integers, m and r are relatively prime, and $r > 0$. Find $m + n + r$.

Solution. In general, in order to solve a polynomial we'll need to factor it completely, or rewrite it in some clever form (*e.g.* $(3x^2 - 1)^2 = (2x^2 - 1)^2$; can you see why this is useful?). Given the answer form, the problem seems to suggest that we need a quadratic factor of the polynomial. But the polynomial seems extremely hard to factor at first glance; there are no rational roots (since there are only two roots) and no obvious factors. So let's try grouping the coefficients and factoring them; maybe we can extract a common factor this way. We note that the middle three terms form a geometric sequence:

$$x + 10x^3 + 100x^5 = x \cdot \frac{1000x^6 - 1}{10x^2 - 1}.$$

This is great, because the other two terms have a factor of $1000x^6 - 1:$

$$2000x^6 - 2 = 2(1000x^6 - 1)$$

so

$$2000x^6+100x^5+10x^3+x-2 = (1000x^6-1)\left(2+\frac{x}{10x^2-1}\right) = \frac{(1000x^6-1)(20x^2+x-2)}{10x^2-1}.$$

This can be rewritten as

$$(100x^4 + 10x^2 + 1)(20x^2 + x - 2)$$

so the two real roots of the polynomial come from the second factor, which are

$$\frac{-1 \pm \sqrt{1^2 + 4 \cdot 20 \cdot 2}}{40} = \frac{-1 \pm \sqrt{161}}{40}.$$

So the answer is $-1 + 161 + 40 = 200$. □

§3 Combining Equations

When faced with systems of equations, it can be useful to consider all of the equations as a whole. Instead of focusing on each equation individually and trying to solve it, we can try combining the given equations in a clever way that either directly gives us an answer or creates cleaner equations to solve.

Example 3.1 (1990 AIME 15)
If a, b, x and y are real numbers such that $ax + by = 3$, $ax^2 + by^2 = 7$, $ax^3 + by^3 = 16$, and $ax^4 + by^4 = 42$, find $ax^5 + by^5$.

Solution. We should first ask ourselves how we are going to get the desired answer. In this case, multiplying the equations we have may seem like a good idea since we can easily get x^5, however we also get a coefficient of a^2 which is hard to convert to an a. Instead, since we are given a "sequence" of equations of the form $ax^n + by^n$, we should think about how to get from $ax^n + by^n$ to $ax^{n+1} + by^{n+1}$. Naively, we can try to multiply the first expression by $(x + y)$ and see what we get. This gives

$$(x + y)(ax^n + by^n) = (ax^{n+1} + by^{n+1}) + xy(ax^{n-1} + by^{n-1})$$

This is really good, since we know a lot of values of $ax^n + bx^n$, so we should be able to set up a system to find $x + y$ and xy. In particular, substituting in $n = 2, 3$ gives

$$7(x + y) = 16 + 3xy \quad \text{and} \quad 16(x + y) = 42 + 7xy$$

Solving, we get $x + y = -14$, $xy = -38$. So,

$$ax^5 + bx^5 = (x + y)(ax^4 + bx^4) - xy(ax^3 + bx^3) = -14 \cdot 42 + 38 \cdot 16 = 20$$

□

Example 3.2 (Mildorf Mock AIME)

a, b,and c are complex numbers such that

$$a + b + c = 1$$
$$a^2 + b^2 + c^2 = 3$$
$$a^3 + b^3 + c^3 = 7$$

Find $a^7 + b^7 + c^7$.

Solution. This ideas in solution should seem familiar to the solution of Example 3.1. Let $S_n = a^n + b^n + c^n$. The first step is to notice that S_y is going to be hard to calculate by manipulating the given equations. We need some sort of general manipulation we can use to get to S_7. Therefore, the idea is took for some sort of recursion for S_n. We are given the initial terms S_1, S_2, and S_3, so we can calculate S_7 if we have a recursion.

To create a term of the form a^{n+1}, we want to multiply a^n by a, so one common idea may be to multiply $a^n + b^n + c^n$ by $a + b + c$. We obtain,

$$S_n(a + b + c) = S_{n+1} + (a^n b + a^n c + b^n a + b^n c + c^n a + c^n b)$$

We want to somehow express the second term in terms of previous S_i. Then, we notice that we will get terms of that form if we multiply S_{n-1} by $ab + bc + ca$. Therefore, we get from expansion

$$\begin{aligned}(ab + bc + ca)S_{n-1} &= (a^n b + a^n c + b^n a + b^n c + c^n a + c^n b) + a^{n-1}bc + b^{n-1}ac + c^{n-1}ab \\ &= (a^n b + a^n c + b^n a + b^n c + c^n a + c^n b) + abcS_{n-2}\end{aligned}$$

Thus, we have obtained $(a^n b + a^n c + b^n a + b^n c + c^n a + c^n b)$ in terms of S_{n-1} and S_{n-2}, so we can substitute this back into the first equation.

$$S_n(a + b + c) = S_{n+1} + (ab + bc + ca)S_{n-1} - abcS_{n-2}$$

$$S_{n+1} = (a + b + c)S_n - (ab + bc + ca)S_{n-1} + abcS_{n-2}$$

Now, our recursion is almost complete. We just have to determine the values of $ab+bc+ca$ and abc. The former is quite simple. Note that

$$ab + bc + ca = \frac{(a + b + c)^2 - (a^2 + b^2 + c^2)}{2} = \frac{1 - 3}{2} = -1$$

For the latter, we can use the following factorization:

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

Substituting what we know,

$$7 - 3abc = 4 \implies abc = 1.$$

Now, our recursion becomes $S_{n+1} = S_n + S_{n-1} + S_{n-2}$. From the given values, we have $S_4 = 11$, $S_5 = 21$, $S_6 = 39$, and $S_7 = 71$.

□

Example 3.3 (2014 HMMT Algebra 9)

Given that a, b, c are complex numbers satisfying

$$a^2 + ab + b^2 = 1 + i$$
$$b^2 + bc + c^2 = -2$$
$$c^2 + ca + a^2 = 1$$

compute $(ab + ac + bc)^2$.

Solution. Like the previous question, we should note that we are asked to find $(ab+ac+bc)^2$ instead of a, b, c. What this most likely means is that a, b, c are quite difficult to find. Expanding, we see that we want to find

$$(ab + ac + bc)^2 = a^2b^2 + a^2c^2 + b^2c^2 + 2a^2bc + 2ab^2c + 2abc^2$$

The easiest way to get an expression with the a^2bc terms is to just multiply two of the expressions we are given. For example,

$$(b^2 + bc + c^2)(c^2 + ca + a^2) = a^2b^2 + a^2c^2 + b^2c^2 + c^4 + bc^3 + ac^3 + (a^2bc + ab^2c + abc^2)$$

Recall, though, that we want to preserve symmetry, since both our expressions, and our desired result are symmetrical. We can do this by adding this result with our other two similar products. Namely,

$$(a^2 + ab + b^2)(a^2 + ac + c^2) + (a^2 + ab + b^2)(b^2 + bc + c^2) + (a^2 + ac + c^2)(b^2 + bc + c^2)$$

$$= (a^4+b^4+c^4)+3(a^2b^2+a^2c^2+b^2c^2)+(a^3b+b^3a+a^3c+c^3a+b^3c+c^3b)+3(a^2bc+ab^2c+abc^2)$$

Now, let's try to address the annoying a^4 and a^3b terms. Note that these terms all appear in

$$(a^2 + ab + b^2)^2 = a^4 + b^4 + 2a^3b + 2ab^3 + 3a^2b^2$$

So, if we try a similar symmetric sum as before, note that

$$(a^2 + ab + b^2)^2 + (a^2 + ac + c^2)^2 + (b^2 + bc + c^2)^2$$

$$= 2(a^4 + b^4 + c^4) + 2(a^3b + b^3a + a^3c + c^3a + b^3c + c^3b) + 3(a^2b^2 + a^2c^2 + b^2c^2)$$

Aha! The coefficients work out such that if I multiply the first expression by 2 and subtract the second expression, I eliminate everything I don't want! Doing this, we get

$$2\sum(a^2 + ab + b^2)(a^2 + ac + c^2) - \sum(a^2 + ab + b^2)^2$$

$$= 3(a^2b^2 + a^2c^2 + b^2c^2) + 6(a^2bc + ab^2c + abc^2) = 3(ab + ac + bc)^2$$

So, our final answer is

$$\frac{2((1 + i)(-2) + (1 + i)(1) + (-2)(1)) - ((1 + i)^2 + (-2)^2 + (1)^2)}{3} = -\frac{11}{3} - \frac{4}{3}i$$

□

Note: It looks like the last step was a bit magical, as not only did everything we didn't want cancel out, but we were left with exactly our final answer. Of course, we knew from the beginning this would happen, as this is a contest question, and thus there must exist a certain level of artificiality in order for this question to work, since general systems of quadratics are quite difficult to solve.

In this case, however, there is a deeper idea behind just the random algebraic identity we obtained. For the interested reader, note that $a^2 + ab + b^2$ looks a lot like Law of Cosines when the angle is 120°. See if you can find a geometric interpretation of the question.

§4 More Systems

Example 4.1 (2005 Iberoamerican/1)
Determine all triples of real numbers (x, y, z) such that

$$\begin{aligned} xyz &= 8 \\ x^2y + y^2z + z^2x &= 73 \\ x(y-z)^2 + y(z-x)^2 + z(x-y)^2 &= 98. \end{aligned}$$

Solution. The ugliest expression in this system is the last one, so let's expand it to see if it is simplifiable. Expanding, we get

$$x^2y + y^2z + z^2x + x^2z + y^2x + z^2y - 6xyz = 98$$

Using the other two equations, we get $x^2z + y^2x + z^2y = 98 + 6 \cdot 8 - 73 = 73$. So, we have

$$x^2y + y^2z + z^2x = xy^2 + yz^2 + zx^2 \implies (x-y)(x-z)(y-z) = 0$$

As the equations are cyclic, we can assume that $x = y$. Now, the first two equations become

$$x^2z = 8, x^3 + x^2z + z^2x = 73$$

The second equation rearranges as $z^2 = \frac{65}{x} - x^2$, and substituting into the first equation yields $65x^3 - x^6 = 64$. This is a quadratic in x^3 which yields

$$x^3 = 1, 64 \implies x = 1, 4$$

So, our solution set is $(1, 1, 8)$, $\left(4, 4, \frac{1}{2}\right)$, and cyclic permutations. □

Note: Though this example is easier than the other ones, it demonstrates the importance of knowing factorizations. After we get $x^2y + y^2z + z^2x = xy^2 + yz^2 + zx^2$, there are ways to show that we must have two of them are equal with inequalities, but you can save yourself a lot of time by quickly recognizing that the expression is actually factorable.

§5 A Tricky Example

Example 5.1 (2014 AIME 14)
Let m be the largest real solution to the equation

$$\frac{3}{x-3}+\frac{5}{x-5}+\frac{17}{x-17}+\frac{19}{x-19}=x^2-11x-4$$

There are positive integers a, b, and c such that $m=a+\sqrt{b+\sqrt{c}}$. Find $a+b+c$.

Solution. The key to this problem is to try to preserve symmetry for as long as possible. First, note that adding 1 to each of the fractions on the left makes them a lot simpler, so let's break up the -4 on the RHS to do that. We get

$$\frac{x}{x-3}+\frac{x}{x-5}+\frac{x}{x-17}+\frac{x}{x-19}=x(x-11)$$

Zero definitely isn't going to be the largest real solution to this equation, so we can safely divide by x as well. Now, the resultant expression is a quintic, which is unfortunately unsolvable generally. However, in this case, we have symmetry on our side. Note that both the LHS and RHS are "centered" around $x-11$. So, if we make the substitution $y=x-11$, our equation becomes

$$\frac{1}{y+8}+\frac{1}{y+6}+\frac{1}{y-6}+\frac{1}{y-8}=y$$

Now, if we group together similar terms on the left, we get

$$\frac{2y}{y^2-64}+\frac{2y}{y^2-36}=y \implies \frac{1}{y^2-64}+\frac{1}{y^2-36}=\frac{1}{2}$$

This is a quadratic now, which we can solve, but let's continue our symmetry argument. The LHS is now symmetric around y^2-50, so if we let $z=y^2-50$, we get

$$\frac{1}{2}=\frac{1}{z-14}+\frac{1}{z+14} \implies z^2-196=4z$$

Quadratic formula gives $z=2+\sqrt{200}$, so back substituting gives $y=\sqrt{52+\sqrt{200}} \implies x=11+\sqrt{52+\sqrt{200}}$. So, our answer is $200+52+11=263$. □

§6 Problems

Problem 6.1 (2020 CMIMC Algebra #1)**.** Suppose x is a real number such that $x^2 = 10x + 7$. Find the unique ordered pair of integers (m, n) such that $x^3 = mx + n$.

Problem 6.2 (Brilliant)**.** Suppose that a and b are real numbers such that $ab = 3$ and $a^2 + b^2 = 8$. Find the value of $\frac{a^7 - ab^6 - a^6 b + b^7}{a+b}$.

Problem 6.3 (2000 AIME I #7)**.** Suppose that x, y, and z are three positive numbers that satisfy the equations $xyz = 1$, $x + \frac{1}{z} = 5$, and $y + \frac{1}{x} = 29$. Then $z + \frac{1}{y} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 6.4 (2010 AIME I #9)**.** Let (a, b, c) be the real solution of the system of equations $x^3 - xyz = 2$, $y^3 - xyz = 6$, $z^3 - xyz = 20$. What is the greatest possible value of $a^3 + b^3 + c^3$?

Problem 6.5 (2018 CMIMC Algebra #5)**.** Suppose that a,b, and c are nonzero real numbers such that

$$bc + \frac{1}{a} = ca + \frac{2}{b} = ab + \frac{7}{c} = \frac{1}{a+b+c}.$$

Find $a + b + c$.

Problem 6.6 (1989 AIME #8)**.** Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123. \end{aligned}$$

Find the value of $16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7$.

Problem 6.7 (AoPS)**.** Let α satisfy $\alpha^3 - \alpha^2 - 2\alpha + 1 = 0$. If $\alpha = x + \frac{1}{x}$, then find

$$x^{64} - 2x^{52} + 3x^{43} + 2x^{38} - 2x^{29} + 5x^{17} + 5x^{10} - 7x^7 + 7.$$

Problem 6.8 (2020 CMIMC Algebra #8)**.** Compute the positive difference between the two real solutions to the equation

$$(x-1)(x-4)(x-2)(x-8)(x-5)(x-7) + 48\sqrt{3} = 0.$$

Problem 6.9. Let a, b, c, and d be complex numbers such that $a + b + c + d = 1$, $ab + bc + cd + da + ac + bd = 1$, $abc + acd + abd + bcd = 2$, and $abcd = 3$. Find the value of $a^6 + b^6 + c^6 + d^6$.

Problem 6.10 (AoPS)**.** Let a, b, c be the real roots of $x^3 - 4x^2 - 32x + 17 = 0$. Solve for x in

$$\sqrt[3]{x-a} + \sqrt[3]{x-b} + \sqrt[3]{x-c} = 0.$$

Problem 6.11 (2010 HMMT Algebra #5)**.** Suppose that x and y are complex numbers such that $x + y = 1$ and $x^{20} + y^{20} = 20$. Find the sum of all possible values of $x^2 + y^2$.

Problem 6.12 (2019 CMIMC Algebra #6)**.** Let a,b, and c be the distinct solutions to the equation $x^3 - 2x^2 + 3x - 4 = 0$. Find the value of

$$\frac{1}{a(b^2 + c^2 - a^2)} + \frac{1}{b(c^2 + a^2 - b^2)} + \frac{1}{c(a^2 + b^2 - c^2)}.$$

Problem 6.13 (2010 HMMT Algebra #7)**.** Let x, y, z, a, b, c be nonzero complex numbers such that $x + y + z = 67$ and $xy + yz + xz = 2010$. Given that

$$\frac{b+c}{x-3} = a, \qquad \frac{c+a}{y-3} = b, \qquad \frac{a+b}{z-3} = c,$$

Find the value of xyz.

Problem 6.14 (2013 HMMT Algebra #8)**.** Let x, y be complex numbers such that $\frac{x^2+y^2}{x+y} = 4$ and $\frac{x^4+y^4}{x^3+y^3} = 2$. Find all possible values of $\frac{x^6+y^6}{x^5+y^5}$.

Problem 6.15 (EGMO 2019/1)**.** Find all triples (a, b, c) of real numbers such that $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b.$$

Problem 6.16 (Mexico 2011/3)**.** Let n be a positive integer. Find all real solutions $(a_1, a_2, \ldots, a_n)$ to the system:

$$\begin{aligned} a_1^2 + a_1 - 1 &= a_2 \\ a_2^2 + a_2 - 1 &= a_3 \\ &\vdots \\ a_n^2 + a_n - 1 &= a_1 \end{aligned}$$

Problem 6.17 (2006 Putnam B1)**.** Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and findits area.

Problem 6.18 (USAMO 2015/1)**.** Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$