Analytic Geometry

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8/26

§1 Cartesian Coordinates

When using 2-dimensional cartesian coordinates, we define the plane by

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

We call (x, y) an *ordered pair*. We choose two perpendicular lines to be the x-axis and the y-axis. On the x-axis and y-axis, $y = 0$ and $x = 0$, respectively. To find the distance between two points in the coordinate plane, we use the distance formula.

Theorem 1.1 (Distance Formula)
The distance between (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Proof. Follows directly from the Pythagorean Theorem. □

A line between two points can be written in the form $ax + by = c$ for some constants a, b, c. We can also solve for y to write it as $y = mx + b$ where m is known as the slope of the line and b is the y-intercept. This is known as slope-intercept form. Slope is arguably the most important notion in coordinate geometry.

Theorem 1.2 (Slope)
The slope of the line between the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Another result that is sometimes useful when given angles is the following.

Theorem 1.3 (Slope with tangent)
If ℓ intersects the x-axis at an angle θ, the slope of ℓ is $\tan\theta$.

Proof. Follows directly from the definition of slope. □

Theorem 1.4 (Parallel and Perpendicular Lines)

Consider two lines ℓ_1 and ℓ_2 not parallel to the coordinate axes with slopes m_1, and m_2, respectively. The lines ℓ_1 and ℓ_2 are *parallel* if and only if $m_1 = m_2$. Additionally, the lines ℓ_1 and ℓ_2 are *perpendicular* if and only if $m_1 m_2 = -1$.

Proof. When ℓ_1 and ℓ_2 are parallel, they intersect the x-axis at the same angle, so they have the same slope because the slope is the tangent of the angle. When ℓ_1 intersects the x-axis at an angle θ_1 and ℓ_2 intersects the x-axis at an angle θ_2, for the lines to be perpendicular, we have $\theta_2 = \theta_1 + 90^\circ$, which is equivalent to $\tan\theta_1 \tan\theta_2 = -1$. □

Sometimes, it is useful to find the distance between a point and a line.

Theorem 1.5 (Distance between Point and Line)

The distance between the point (x_0, y_0) and the line $ax + by + c = 0$ is

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

We will not show the proof here, but the easiest way to show this is with vectors.

Finally, we need to know how to calculate area of a polygon from the coordinates.

Theorem 1.6 (Shoelace Formula)

Suppose the polygon P has vertices (a_1, b_1), (a_2, b_2), ... , (a_n, b_n), listed in clockwise order. Then the area of P is

$$\frac{1}{2}|(a_1b_2 + a_2b_3 + \cdots + a_nb_1) - (b_1a_2 + b_2a_3 + \cdots + b_na_1)|$$

Proof. We can show this is true for a triangle ($n = 3$) and use induction to show the result for polygons of all sizes. □

Remark 1.7. The Shoelace Theorem gets its name because if one lists the coordinates in a column,

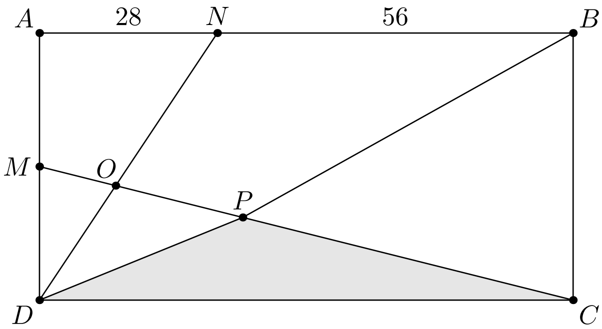
$$\begin{gathered}(a_1, b_1)\\(a_2, b_2)\\\vdots\\(a_n, b_n)\\(a_1, b_1)\end{gathered}$$

and marks the pairs of coordinates to be multiplied, the resulting image looks like laced-up shoes.

Generally, we use Cartesian coordinates when we can easily write down the coordinates of most points and there are many parallel/perpendicular lines, so we can use the results above. Sometimes, the choice of the origin is very important and can simplify calculations greatly as we will see in the examples below.

Example 1.8 (2017 AIME II # 10)

Rectangle $ABCD$ has side lengths $AB = 84$ and $AD = 42$. Point M is the midpoint of $\overline{AD}$, point N is the trisection point of $\overline{AB}$ closer to A, and point O is the intersection of $\overline{CM}$ and $\overline{DN}$. Point P lies on the quadrilateral $BCON$, and $\overline{BP}$ bisects the area of $BCON$. Find the area of $\triangle CDP$.



Solution. Since the diagram involves a rectangle and the intersections of easily computable lines, we can solve with the use of coordinates. Let $D = (0, 0)$, $C = (84, 0)$, $B = (84, 42)$, and $A = (0, 42)$. Then, we can see that point N is $(28, 42)$ and $M = (0, 21)$. The first order of business is to calculate the coordinates of point O. To do this, we will find the equations of lines DN and CM. From the coordinates that we calculated above, we can find that DN has equation $y = \frac{3}{2}x$ and CM has equation $y = -\frac{1}{4}x + 21$. Setting both of these equations equal to each other, we obtain,

$$\frac{3}{2}x = -\frac{1}{4}x + 21 \implies x = 12.$$

If we plug into either of the lines, we find $y = 18$, so $O = (12, 18)$.

Now, we are given that $[BPC] = \frac{1}{2}[BCON]$. To find the coordinates of P from this, we need to find the area of $BCON$, which can be done either with the Shoelace Theorem, or simply with one-half base times height. We see

$$[BCON] = [ABCD] - [AND] - [COD] = 84 \cdot 42 - \frac{1}{2} \cdot 42 \cdot 28 - \frac{1}{2} \cdot 84 \cdot 18 = 2184$$

Therefore, $[BPC] = \frac{1}{2}[BCON] = 1092$. Now, since $BC = 42$, the height from P to BC is $\frac{2 \cdot 1092}{42} = 52$. Since the x coordinate of B or C is 84, the x coordinate of P is $84 - 52 = 32$.

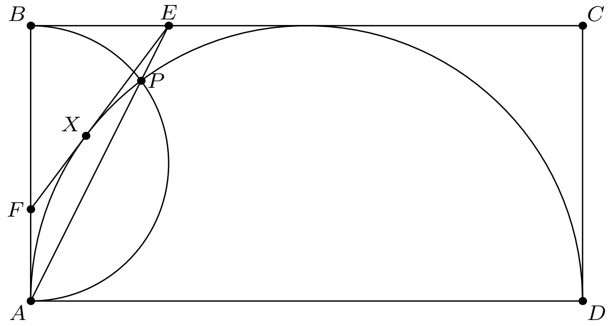
Now, this is enough to find the y-coordinate of P because we know that P lies on CM with equation $y = -\frac{1}{4}x + 21$. Plugging in the value of x, we see $y = 13$. Therefore, the height from P to CD is 13, and the base $CD = 84$. The final answer is then

$$[CPD] = \frac{1}{2} \cdot 13 \cdot 84 = \boxed{546}.$$

□

Example 1.9 (2018 PUMaC)

Consider rectangle $ABCD$ with $AB = 30$ and $BC = 60$. Construct circle T whose diameter is AD. Construct circle S whose diameter is AB. Let circles T and S intersect at P, so that $P \neq A$. Let AP intersect BC at E. Let F be the point on AB so that EF is tangent to the circle with diameter AD. Find the area of triangle AEF.



Solution. Let AD and AB be the positive $x-$ and $y-$ axes, respectively. Thus, we find that A is at $(0,0)$, B is at $(0,30)$, C is at $(60,30)$, and D is at $(60,0)$. Now, we wish to find the equations of the circles. We first consider circle T. Note that its center is the midpoint of AD and its radius is $\frac{1}{2}AD$. Thus, we find that its equation is

$$(x-30)^2+y^2=900$$

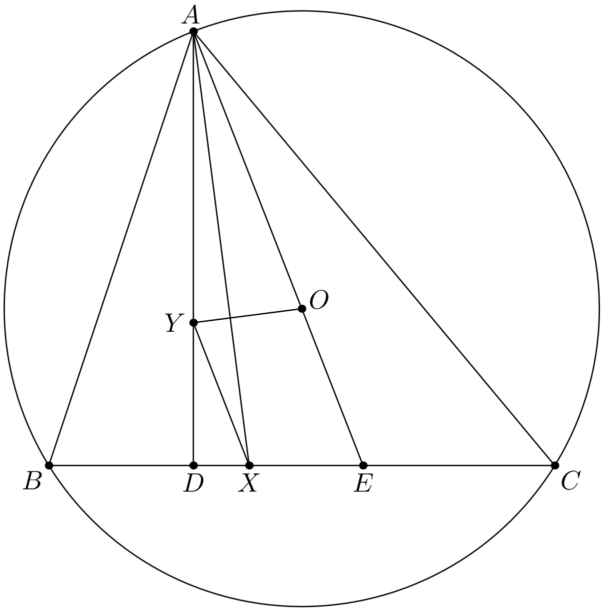
. Similarly, we have that the equation of circle S is

$$x^2+(y-15)^2=225$$

. Now, we wish to solve this system of equations. We note that we may subtract the two equations to remove the x^2+y^2 terms, giving us $60x-30y=0$, so $y=2x$. Now, plugging this back into the equation for circle T gives us $5x^2-60x=0$, so we find $x=0,12$, corresponding with points A and P. Thus, we find that the coordinates of point P are $(12,24)$. Now, we see that the equation for AP is $y=2x$. We see that this intersects BC, which has equation $y=30$, at $(15,30)$. Thus, we find that E is at $(15,30)$. Now, let EF be tangent to T at X. Note that the length of one of the tangents from E to T, the one along BC, has length 15, so the other tangent must as well. Thus, we find $EX=15$. Thus, we have that X must lie on the circle $(x-15)^2+(y-30)^2=225$, as that is the circle with center E and radius 15. Now, we intersect this equation with that of circle T, which we recall is $(x-30)^2+y^2=225$. Subtracting these equations gives $30x-60y+900=0$, so $x=2y-30$. Substituting this back into the equation for circle T gives us $(2y-60)^2+y^2=900$, or $5y^2-240y+2700=0$. Note that this factors as $5(y-18)(y-30)=0$, so we get $y=18,30$. Note that $y=18$ corresponds with X since $y=30$ corresponds with the tangent from E along BC. Thus, we find that X is at $(6,18)$, so that the equation of EX is $y=\frac{4}{3}x+10$. Thus, F is at $(0,10)$. Now, note that we can find the area of AEF since its base AF has length 10 while its height has length 15. Thus, we have the area is $\frac{1}{2}\cdot 10\cdot 15=\boxed{75}$. □

Example 1.10 (2014 HMMT)

Let ABC be an acute triangle with circumcenter O such that $AB=4$, $AC=5$, $BC=6$. Let D be the foot of the altitude from A to $\overline{BC}$ and E be the intersection of lines AO and BC. Suppose that X is on $\overline{BC}$ between D and E such that there is a point Y on $\overline{AD}$ satisfying $\overline{XY}\parallel\overline{AO}$ and $\overline{YO}\perp\overline{AX}$. Determine the length of BX.



Solution. To solve this problem with coordinates, we need to think carefully about where to place the origin. Since our end goal is to compute BX, it would make sense to let BC be the x axis. Since many of the lines involved in this question pass through A such as AO and AX, we could put A on the y-axis to ease computations as well. This would make D the origin. Our next step is to find the coordinates of the other points marked in the diagram, which is mostly routine.

By Heron's formula, $[ABC] = \frac{15\sqrt{7}}{4}$, so $AD = \frac{2[ABC]}{BC} = \frac{5\sqrt{7}}{4}$. Therefore, $A = (0, \frac{5\sqrt{7}}{4})$. Now, from the Pythagorean theorem on ABD, we see $BD = \frac{9}{4}$. Therefore, we see $B = (-\frac{9}{4}, 0)$ and $C = (\frac{15}{4}, 0)$. Notice that point E is not really relevant in this problem because all the conditions are in terms of point O. Therefore, we will not find the coordinates of E. For the coordinates of O, note that the perpendicular bisector of BC passes through O, so the x-coordinate must be $3 - \frac{9}{4} = \frac{3}{4}$. To find the y coordinate, we can use the distance formula. We know the distance from O to B is the cirumradius which is

$$R = \frac{abc}{4[ABC]} = \frac{4 \cdot 5 \cdot 6}{15\sqrt{7}} = \frac{8\sqrt{7}}{7}$$

From the distance formula, $\sqrt{3^2 + y^2} = R$, and we solve to find $y = \frac{\sqrt{7}}{7}$. Therefore, the coordinates of O are $(\frac{3}{4}, \frac{\sqrt{7}}{7})$.

Now, that we have computed all we can from the diagram so far, we can set a variable for X. Let $X = (a, 0)$. First, we have to find the coordinates of Y in terms of a. To do this, we will use the slope condition $\overline{XY} \parallel \overline{AO}$. The slope of AO is

$$\frac{\frac{\sqrt{7}}{7} - \frac{5\sqrt{7}}{4}}{\frac{3}{4}} = -\frac{31\sqrt{7}}{21}.$$

Since the slope of XY is the same and Y is on the y-axis, we find $Y = (0, \frac{31\sqrt{7}}{21}a)$.

To solve for a, we will use the condition $\overline{YO} \perp \overline{AX}$. The slope of $\overline{YO}$ is

$$\frac{\frac{\sqrt{7}}{7} - \frac{31\sqrt{7}}{21}a}{\frac{3}{4} - 0} = \frac{4\sqrt{7}}{63}(3 - 31a)$$

The slope of $\overline{AX}$ is

$$\frac{0 - \frac{5\sqrt{7}}{4}}{a - 0} = -\frac{5\sqrt{7}}{4a}$$

Since $\overline{YO} \perp \overline{AX}$, the product of the slopes is -1.

$$\frac{4\sqrt{7}}{63}(3-31a) \cdot -\frac{5\sqrt{7}}{4a} = -1$$

Simplifying,

$$\frac{5}{9a}(3-31a) = 1 \implies 9a = 15 - 155a.$$

Solving, we get $a = \frac{15}{164}$. Therefore, $BX = \frac{9}{4} + \frac{15}{164} = \boxed{\frac{96}{41}}$. □

> **Remark 1.11.** Alternatively, with angle chasing, we can prove that AX is a symmedian (reflection of the median across the angle bisector). We can use well-known properties of symmedians to finish. However, this method requires much more geometric knowledge than our coordinate approach.

The most important idea in the previous problem was setting up Cartesian coordinates and choosing the origin. With a different choice of origin, the computations could be much more involved. In general, we want to stress to importance of bashing intelligently as it will simplify a lot of the work.

§2 Complex Numbers

We will also consider the use of complex numbers as an alternate coordinate system. Instead of using an x-coordinate and a y-coordinate, we will represent a point using the real part and the imaginary part of a complex number. The primary use of complex numbers instead of Cartesian coordinates in computational problems is the fact that it is easy to model *rotations* with them. We'll use some facts about complex numbers mentioned in our discussion of the algebraic properties of complex numbers in Lesson 7.

Theorem 2.1

If a point Z with coordinate z is rotated an angle θ counterclockwise about the origin, then its image is $(\cos\theta + i\sin\theta)z$.

Proof. This follows from the fact that when we multiply two complex numbers, their magnitudes multiply and their arguments add. Here, $\cos\theta + i\sin\theta$ has absolute value 1 so $(\cos\theta + i\sin\theta)z$ has the same magnitude as z. It's argument is also θ greater than that of z, so it is indeed the image of z under the rotation. □

Using a suitable translation, we can rotate about points other than the origin.

Theorem 2.2

If a point Z with coordinate z is rotated an angle θ counterclockwise about a point A with coordinate a, then its image is $(\cos\theta + i\sin\theta)(z-a) + a$.

Proof. Let's shift A to the origin, so that $a' = 0$ and the image of Z is $z' = z - a$. If we rotate Z' an angle θ around the origin, then shift back by a, we'll get the image of Z under rotation about A. The result is

$$(\cos\theta + i\sin\theta)z' + a = (\cos\theta + i\sin\theta)(z-a) + a$$

as desired. □

Note that since we can easily transfer between complex numbers and Cartesian coordinates, we can also express these rotation formulas in a Cartesian coordinate system. For example, the image of (x, y) under a clockwise rotation of θ around the origin is

$$(x, y) \mapsto (x\cos\theta - y\sin\theta, x\cos\theta + y\sin\theta).$$

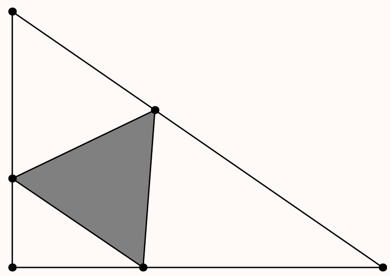
While complex numbers are not technically necessary for this, they provide a simple way to see why this should be true. Indeed, in complex numbers we can multiply

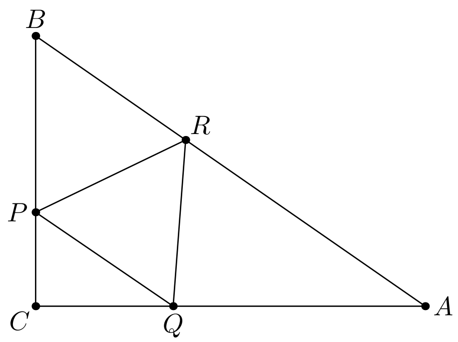
$$(x + yi)(\cos\theta + i\sin\theta)$$

and derive the formula above.

Example 2.3 (2017 AIME I # 15)

The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}$, 5, and $\sqrt{37}$, as shown, is $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$.





Solution. Equilateral triangles are a classic example of using rotation formulas. While we won't use any complex numbers (we'll use Cartesian coordinates instead), keep in mind that rotation formulas are most naturally derived from complex numbers.

Since we have a right triangle, let's set $C = (0, 0)$, $A = (5, 0)$, and $B = (0, 2\sqrt{3})$. We know that P and Q lie on the axes, so we can set $P = (0, p)$ and $Q = (q, 0)$. For this choice of P and Q, we need to find a point R such that triangle PQR is equilateral and R is on side AB.

We see that to get R, we can rotate Q sixty degrees counterclockwise around P (if the rotation was clockwise, PQR would be equilateral but R would not lie in the first quadrant). How can we do this? First, let's translate P to the origin:

$$P' = (0, 0), Q' = (q, -p).$$

We now use the rotation formula to rotate Q'; since $\cos(60°) = \frac{1}{2}$ and $\sin(60°) = \frac{\sqrt{3}}{2}$ the image is

$$\left(\frac{1}{2}q + \frac{\sqrt{3}}{2}p, \frac{\sqrt{3}}{2}q - \frac{1}{2}p\right).$$

Finally, we translate Q back to $(0, p)$ by adding p to the x-coordinate:

$$\left(\frac{1}{2}q + \frac{\sqrt{3}}{2}p, \frac{\sqrt{3}}{2}q + \frac{1}{2}p\right).$$

Now that we have an expression for R, we need it to lie on side AB. Check that the equation of line AB is $2\sqrt{3}x + 5y = 10\sqrt{3}$. Plugging this in, we get

$$2\sqrt{3}\left(\frac{1}{2}q + \frac{\sqrt{3}}{2}p\right) + 5\left(\frac{\sqrt{3}}{2}q + \frac{1}{2}p\right) = 10\sqrt{3}.$$

This simplifies to

$$11p + 7\sqrt{3}q = 20\sqrt{3}.$$

Now that we have a constraint on p and q, let's find the area of the equilateral triangle. By the Pythagorean Theorem $PQ = \sqrt{p^2 + q^2}$ so the area is

$$\frac{\sqrt{3}}{4}(p^2 + q^2).$$

So we want to minimize $p^2 + q^2$ subject to the constraint $11p + 7\sqrt{3}q = 20\sqrt{3}$. One can do this by solving the linear equation for q in terms of p, and then plugging it in to obtain a quadratic in terms of p, which can be maximized. A cleaner way is to use the Cauchy-Schwarz inequality:

$$(p^2 + q^2)(11^2 + (7\sqrt{3})^2) \geq (11p + 7\sqrt{3}q)^2 = 1200 \implies p^2 + q^2 \geq \frac{300}{67}$$

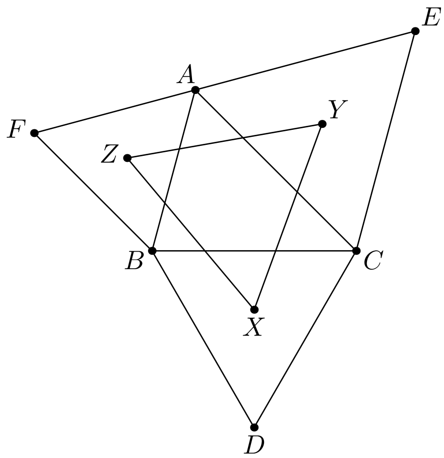
Equality can be achieved when we have the ratio $p : q = 11 : 7\sqrt{3}$. Hence the minimal area is

$$\frac{\sqrt{3}}{4} \cdot \frac{300}{67} = \frac{75\sqrt{3}}{67}$$

and the answer is $75 + 3 + 67 = \boxed{145}$. □

Example 2.4 (Napoleon's Theorem)

Let ABC be a triangle. Points D, E, F satisfy that triangles BCD, CAE, and ABF are all equilateral and don't intersect the interior of triangle ABC. Show that the centers of these three equilateral triangles form an equilateral triangle.



Solution. This time, we will be doing everything in complex numbers, using a lowercase letter to denote the complex number of the point labeled with the corresponding uppercase letter. Let's try to express all the points in the diagram in terms of a, b, and c.

Looking at D, we see that it is the image of B under a counterclockwise rotation of 60° about C. Thus

$$d = (\cos(60^\circ) + i\sin(60^\circ))(b - c) + c = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) b + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) c$$

Since x is the centroid of $\triangle BCD$, we have that

$$x = \frac{b + c + d}{3} = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) b + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) c.$$

Through similar calculations (or just permuting the variables), we get

$$y = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) c + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) a, \ z = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) a + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) b.$$

Now that we have the coordinates of x, y, and z, we just have to show that the triangle they form is equilateral. In order to do this, we can show that if we rotate Y sixty degrees counterclockwise about X, then we get Z. In algebraic terms, we need to show this is

$$z = (\cos(60^\circ) + i\sin(60^\circ))(y - x) + x.$$

We know what x, y, z are in terms of a, b, c, so we can jump into the calculation. We have

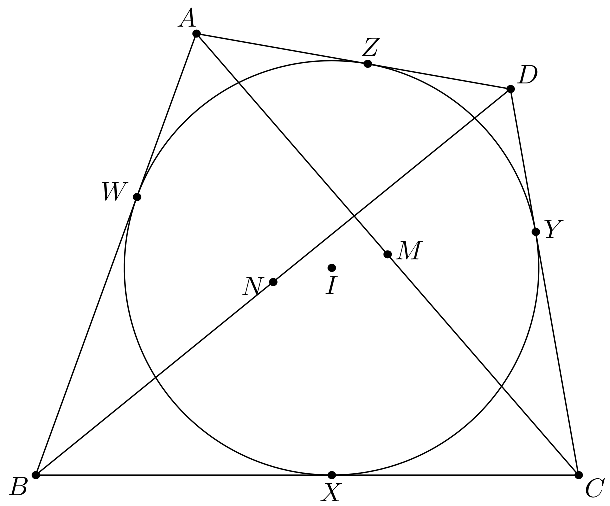
$$\begin{aligned}
&(\cos(60^\circ + i\sin(60^\circ)(y - x) + x \\
&= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) c + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) a - \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) b - \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) c\right) + x \\
&= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) a - \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) b + \frac{\sqrt{3}}{3}ic\right) + x \\
&= \left(\frac{1}{2} + \frac{\sqrt{3}}{3}i\right) a - \frac{\sqrt{3}}{3}ib + \left(-\frac{1}{2} + \frac{\sqrt{3}}{6}i\right) c + x \\
&= \left(\frac{1}{2} + \frac{\sqrt{3}}{3}i\right) a + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right) b
\end{aligned}$$

which is precisely z. So Z is the image of Y under a 60° counterclockwise rotation around X, so we can conclude that XYZ is an equilateral triangle. □

Complex numbers are extremely useful in olympiad geometry problems, providing a coordinate system that is nearly always superior to Cartesian coordinates. This is due to special properties of complex numbers when they lie on the *unit circle*, that is, the circle centered at the origin with radius 1. (This property is that if $|z| = 1$ then $\overline{z} = \frac{1}{z}$.) The following problem is a glimpse of some of the power of complex numbers in olympiad geometry.

Example 2.5 (2017 HMMT Geometry # 10)

Let $ABCD$ be a quadrilateral with an inscribed circle ω. Let I be the center of ω, and let $IA = 12$, $IB = 16$, $IC = 14$, and $ID = 11$. Let M be the midpoint of segment AC. Compute $\frac{IM}{IN}$, where N is the midpoint of segment BD.



Solution. First, note that what we want to find is a ratio. Thus, it's okay if we rescale the quadrilateral so that its inradius is 1. Then there is a real number k for which

$$IA = 12k,\ IB = 16k,\ IC = 14k,\ ID = 11k.$$

Suppose that the incircle (which we set to be the unit circle) touches the sides at W, X, Y, Z as shown. The reason that we set the incircle to the unit circle is the following lemma:

Lemma 2.6

$a = \frac{2zw}{z+w}$. (This is the tangent intersection formula.)

Proof. Let M be the midpoint of $\overline{WZ}$. We see that M is on $\overline{IA}$, as $IW = IZ$ and $AW = AZ$ (by equal tangents) so $\overline{AI}$ is the perpendicular bisector of WZ. Since $\angle IWA = \angle IZA = 90^\circ$ (by the tangencies) we can use some similar triangles to show that $IM \cdot IA = IW^2 = 1$ (check this). Thus

$$|m| \cdot |a| = 1.$$

Moreover, since I, M, and A are collinear, we have that $\overline{m} \cdot a$ is a real number. This is because the argument of $\overline{m}$ is 360° minus the argument of m, which is 360° minus the argument of a. Then $\overline{m} \cdot a$ has argument $360° = 0°$ and is real. We have

$$|m| \cdot |a| = 1 \implies |\overline{m}| \cdot |a| = 1 \implies |\overline{m} \cdot a| = 1$$

since m and $\overline{m}$ have the same magnitude. But $\overline{m} \cdot a$ has argument zero so it is a positive real. This is enough to imply that

$$\overline{m} \cdot a = 1 \implies a = \frac{1}{\overline{m}}.$$

We now use the properties of the conjugate, including the fact that $\overline{a} = \frac{1}{a}$ and $\overline{b} = \frac{1}{b}$ since a and b have magnitude 1:

$$m = \frac{z+w}{2} \implies \overline{m} = \frac{\overline{z} + \overline{w}}{2} = \frac{\frac{1}{z} + \frac{1}{w}}{2} = \frac{z+w}{2zw}.$$

Thus $a = \frac{1}{\overline{m}} = \frac{2zw}{z+w}$. □

From our lemma, we have that

$$a = \frac{2zw}{z+w}, \ b = \frac{2wx}{w+x}, \ c = \frac{2xy}{x+y}, \ d = \frac{2yz}{y+z}.$$

We now solve for m and n:

$$m = \frac{a+c}{2} = \frac{zw}{z+w} + \frac{xy}{x+y} = \frac{wxy + xyz + yzw + zwx}{(z+w)(x+y)}$$

and similarly $n = \frac{wxy+xyz+yzw+zwx}{(w+x)(y+z)}$. Since I is the origin, we have that $IM = |m|$ and $IN = |n|$, and when we compute the ratio we get a lot of cancellation:

$$\frac{IM}{IN} = \frac{|m|}{|n|} = \left|\frac{m}{n}\right| = \left|\frac{(w+x)(y+z)}{(z+w)(x+y)}\right| = \frac{|w+x||y+z|}{|z+w||x+y|}.$$

To find these quantites, we turn to our given lengths. Since $IA = 12k$, we have

$$12k = IA = |a| = \left|\frac{2zw}{z+w}\right| = \frac{2|z||w|}{|z+w|} = \frac{2}{|z+w|} \implies |z+w| = \frac{2}{12k}.$$

since $|z| = |w| = 1$. Similarly,

$$|w+x| = \frac{2}{16k}, \ |x+y| = \frac{2}{14k}, \ |y+z| = \frac{2}{11k}.$$

Thus, we may toss everything into our final expression to obtain

$$\frac{IM}{IN} = \frac{|w+x||y+z|}{|z+w||x+y|} = \frac{\frac{2}{16k} \cdot \frac{2}{11k}}{\frac{2}{12k} \cdot \frac{2}{14k}} = \boxed{\frac{21}{22}}.$$

□

Remark 2.7. From the diagram, you might notice that I, M, and N are collinear; this result is true in general and is known as *Newton's Theorem.* It can be proved with complex numbers as well, using the complex coordinates we found for M and N. It suffices to show that $\frac{m}{n}$ is real, which can be done by showing that it is equal to its conjugate and using $\overline{z} = \frac{1}{z}$ for z on the unit circle. (Try it!) This method of showing a complex number is real is central to the use of complex numbers in olympiad geometry.

For more on the use of complex numbers in olympiad geometry, check out Evan Chen's handout or chapter 6 of his geometry textbook (EGMO). (Note that this type of complex bashing is solidly an olympiad technique, with little to no use on the AIME.)

§3 Problems

Problem 3.1 (2013 AIME II # 4)**.** In the Cartesian plane let $A = (1, 0)$ and $B = \left(2, 2\sqrt{3}\right)$. Equilateral triangle ABC is constructed so that C lies in the first quadrant. Let $P = (x, y)$ be the center of $\triangle ABC$. Then $x \cdot y$ can be written as $\frac{p\sqrt{q}}{r}$, where p and r are relatively prime positive integers and q is an integer that is not divisible by the square of any prime. Find $p + q + r$.

Problem 3.2 (2011 AIME I # 3)**.** Let L be the line with slope $\frac{5}{12}$ that contains the point $A = (24, -1)$, and let M be the line perpendicular to line L that contains the point $B = (5, 6)$. The original coordinate axes are erased, and line L is made the x-axis, and line M the y-axis. In the new coordinate system, point A is on the positive x-axis, and point B is on the positive y-axis. The point P with coordinates $(-14, 27)$ in the original system has coordinates (α, β) in the new coordinate system. Find $\alpha + \beta$.

Problem 3.3 (1994 AIME # 8)**.** The points $(0, 0)$, $(a, 11)$, and $(b, 37)$ are the vertices of an equilateral triangle. Find the value of ab.

Problem 3.4 (2018 PUMaC Geometry #4)**.** Triangle ABC has $\angle A = 90^o$, $\angle C = 30^o$, and $AC = 12$. Let the circumcircle of this triangle be W. Define D to be the point on arc BC not containing A so that $\angle CAD = 60^o$. Define points E and F to be the foots of the perpendiculars from D to lines AB and AC, respectively. Let J be the intersection of line EF with W, where J is on the minor arc AC. The line D intersects W at H other than at D. Find $[FHJ]$

Problem 3.5 (2017 PUMaC Geometry #5)**.** Rectangle $HOMF$ has $HO = 11$ and $OM = 5$. Triangle ABC has orthocenter H and circumcenter O. M is the midpoint of BC and altitude AF meets BC at F. Find the length of BC.

Problem 3.6 (also Napoleon's Theorem)**.** Let ABC be a triangle. Points D, E, F satisfy that triangles BCD, CAE, and ABF are all equilateral and *intersect* the interior of triangle ABC. Show that the centers of these three equilateral triangles form an equilateral triangle.

Problem 3.7 (2008 AIME II # 13)**.** A regular hexagon with center at the origin in the complex plane has opposite pairs of sides one unit apart. One pair of sides is parallel to the imaginary axis. Let R be the region outside the hexagon, and let $S = \{\frac{1}{z} | z \in R\}$. Then the area of S has the form $a\pi + \sqrt{b}$, where a and b are positive integers. Find $a + b$.

Problem 3.8 (2017 HMMT Geometry #9)**.** Let ABC be a triangle, and let $BCDE$, $CAFG$, $ABHI$ be squares that do not overlap the triangle with centers X, Y, Z respectively. Given that $AX = 6$, $BY = 7$, and $CA = 8$, find the area of triangle XYZ.

Problem 3.9 (2019 HMMT Geometry #9)**.** In a rectangular box $ABCDEFGH$ with edge lengths $AB = AD = 6$ and $AE = 49$, a plane slices through point A and intersects edges BF, FG, GH, HD at points P, Q, R, S respectively. Given that $AP = AS$ and $PQ = QR = RS$, find the area of $APQRS$.

Problem 3.10. Let $ABCD$ be a square and let P be any point inside of it. Prove that the altitudes from A to BP, from B to CP, from C to DP, and from D to AP are concurrent.

Problem 3.11 (2014 Taiwan TST 1/3)**.** Let ABC be a triangle with incenter I, and suppose the incircle is tangent to CA and AB at E and F. Denote by G and H the reflections of E and F over I. Let Q be the intersection of BC with GH, and let M be the midpoint of BC. Prove that IQ and IM are perpendicular.

Problem 3.12 (RMM 2020/1)**.** Let ABC be a triangle with a right angle at C. Let I be the incentre of triangle ABC, and let D be the foot of the altitude from C to AB. The incircle ω of triangle ABC is tangent to sides BC, CA, and AB at A_1, B_1, and C_1, respectively. Let E and F be the reflections of C in lines C_1A_1 and C_1B_1, respectively. Let K and L be the reflections of D in lines C_1A_1 and C_1B_1, respectively.

Prove that the circumcircles of triangles A_1EI, B_1FI, and C_1KL have a common point.