Advanced Geometry

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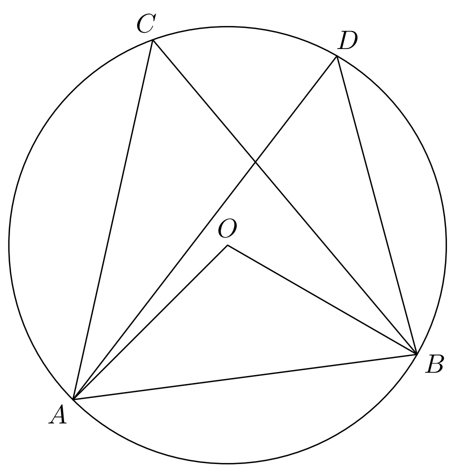
§1 Introduction

In recent years, the AIME has included advanced geometry problems of an "olympiad flavor." These problems, while not necessarily requiring the advanced geometric techniques of olympiads, are qualitatively different from other geometry problems. There will be a lot of geometric work at the beginning of the problem, with the (oftentimes much simpler) computation coming at the end. In these problems, it is *especially* important to make geometric observations before diving into calculation. We'll discuss two methods of making these observations that also see common usage in olympiad geometry, namely angle chasing and radical axes.

§2 Angle Chasing

To angle chase, it is useful to know a few useful theorems.

Theorem 2.1 (Inscribed Angle Theorem)
Let A, B, and C be inscribed in a circle O. Then, $\angle AOB = 2\angle ACB$. Also, angles that subtend the same arc are equal. That is, we have $\angle ACB = \angle ADB$ in the diagram below.



Proof. Let $\angle ACO = \alpha$ and $\angle BCO = \beta$. Then, since $OA = OC$, $\angle OAC = \alpha$ and $\angle AOC = 180° - 2\alpha$. Similarly, $\angle BOC = 180° - 2\beta$. Therefore, we get,

$$\angle AOB = 360° - \angle AOC - \angle BOC = 2(\alpha + \beta) = 2\angle ACB.$$

The second part of the theorem follows directly from the fact that $\angle ACB$ is independent of the position of C. □

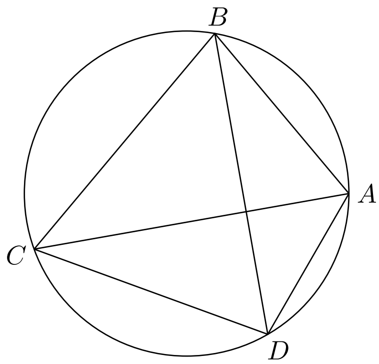
Remark 2.2. It should also be noted that by continuity this result is true even when $D = A$ or $D = B$. In that case, if TA is a tangent to circle O, we have $\angle AOB = 2\angle TAB$ and $\angle TAB = \angle ACB$. The latter result is quite important in problems involving tangents to circles and inscribed angles.

Now, with that in mind, we will move onto the most important concept in angle chasing: cyclic quadrilaterals.

Theorem 2.3 (cyclic quadrilateral)

Let $ABCD$ be a convex quadrilateral. Then, if the points A, B, C, D all lie on one circle, we say $ABCD$ is cyclic. The following three statements are equivalent:

1. $ABCD$ is cyclic
2. $\angle ABC + \angle CDA = 180°$
3. $\angle ABD = \angle ACD$



Proof. The forward direction follows from the Inscribed Angle Theorem, but the converse requires more care. □

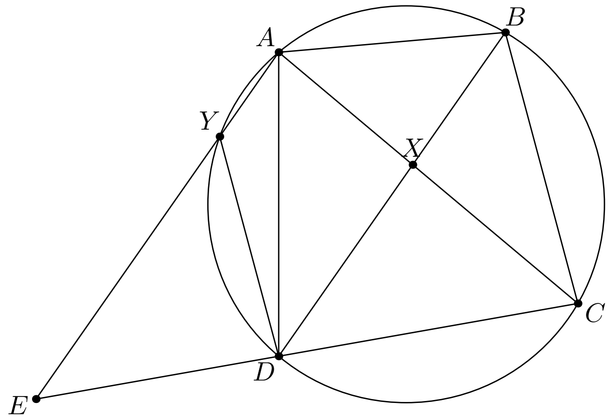
It cannot be stressed enough how important cyclic quadrilaterals are. They will show up in many of the examples below, and they will help to deduce important angle relationships to solve the problems.

Sometimes, cyclic quadrilaterals may be given in the problem when there are 4 points on the same circle. In this case, we can use the angle properties of cyclic quadrilaterals to make other deductions. For example, cyclic quadrilaterals may lead to an important pair of similar triangles, which would help to finish the problem.

Other times, we can prove that four points in our diagram is cyclic from one method, and then use the resulting angle properties to make other deductions about the diagram.

Example 2.4 (2016 PUMaC Geometry #7)

Let $ABCD$ be a cyclic quadrilateral with circumcircle ω and let AC and BD intersect at X. Let the line through A parallel to BD intersect line CD at E and ω at $Y \neq A$. If $AB = 10, AD = 24, XA = 17$, and $XB = 21$, then the area of $\triangle DEY$ can be written in simplest form as $\frac{m}{n}$. Find $m + n$.



Solution. First, using the parallel condition, we can find $\angle DEY = \angle CDB = \angle XAB$. Also, $\angle AYD = \angle ABD$, so $DEY \sim XAB$.

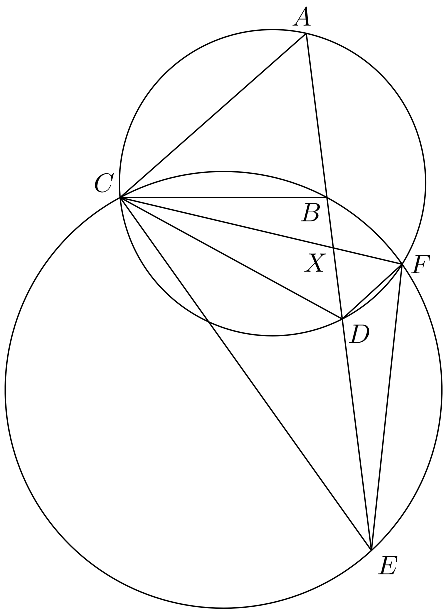
Now, $AY \parallel BD$, so $AYBD$ is an isosceles trapezoid. So, $DY = AB = 10$, and we have

$$[DEY] = \left(\frac{10}{21}\right)^2 [AXB] = \frac{100}{441} \cdot 84 = \frac{400}{21}$$

So, our answer is $400 + 21 = \boxed{421}$. □

Example 2.5 (2019 AIME I #13)

Triangle ABC has side lengths $AB = 4$, $BC = 5$, and $CA = 6$. Points D and E are on ray AB with $AB < AD < AE$. The point $F \neq C$ is a point of intersection of the circumcircles of $\triangle ACD$ and $\triangle EBC$ satisfying $DF = 2$ and $EF = 7$. Then BE can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

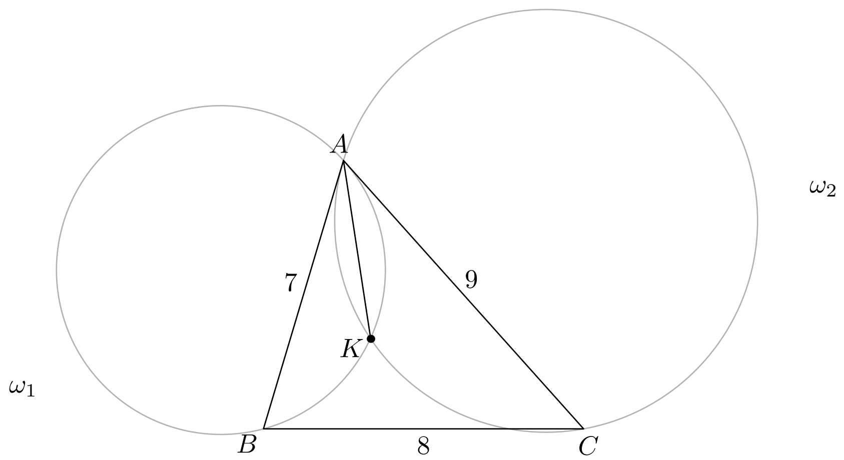


Solution. Notice that $\angle BEF = \angle BCF$ and $\angle ADF = \angle ACF$. Thus, $\angle DFE = \angle ADF - \angle BEF = \angle ACF - \angle BCF = \angle ACB$. Now, by Law of Cosines, we can find $\cos \angle ACB =$

$\frac{5^2+6^2-4^2}{2\cdot5\cdot6} = \frac{3}{4}$. Now, we can find $DE = \sqrt{2^2+7^2-2\cdot2\cdot7\cdot\frac{3}{4}} = 4\sqrt{2}$. Now, let $x = BX$, $y = DX$. Note that we have $ACX \sim FDX$, so $3 = \frac{AC}{DF} = \frac{AX}{FX}$, so $FX = \frac{x+4}{3}$. Now, note that we also have $BCX \sim FEX$, so we have $\frac{7}{5} = \frac{EF}{BC} = \frac{FX}{BX}$, so $FX = \frac{7}{5}x$ as well. Setting these two values equal gives $x = \frac{5}{4}$. Now, note that, by Power of a Point, $AX \cdot DX = CX \cdot FX = BX \cdot EX$, so $(x+4)y = x(y+4\sqrt{2})$, so we can find $y = x\sqrt{2} = \frac{5\sqrt{2}}{4}$. Now, we can find the value of BE to be $x+y+4\sqrt{2} = \frac{5+21\sqrt{2}}{4}$, so our answer is $5+21+2+4 = \boxed{32}$. □

Example 2.6 (2019 AIME II #11)

Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Then $AK = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution. The tangency condition yields $\angle ABK = \angle KAC$ and $\angle KAB = \angle ACK$. Therefore, $\triangle BKA \sim \triangle AKC$. Since $\frac{AB}{AC} = \frac{7}{9}$, the ratio between the two triangles is $\frac{7}{9}$. Let $AK = x$. Then, $\frac{AK}{KC} = \frac{7}{9}$, so $KC = \frac{9}{7}x$. Now, we can also note that

$$\angle AKC = 180^\circ - \angle KAC - \angle KCA = 180^\circ - \angle KAC - \angle BAK = 180^\circ - \angle A.$$

Therefore, we can apply the Law of Cosines on AKC to solve for x. First, we can determine $\cos A$ from the Law of Cosines on $\triangle ABC$. Note that,

$$\cos A = \frac{7^2+9^2-8^2}{2\cdot7\cdot9} = \frac{11}{21}.$$

Therefore, $\cos\angle AKC = -\frac{11}{21}$. From the Law of Cosines on AKC, we find,

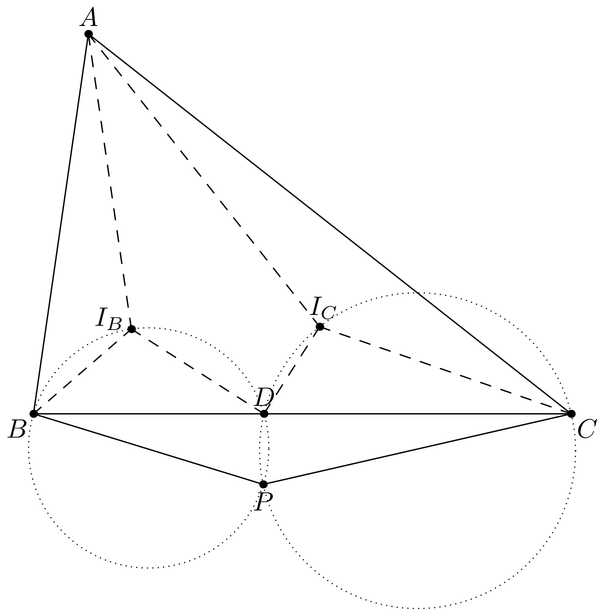
$$x^2 + \frac{81}{49}x^2 + 2x\cdot\frac{9}{7}x\cdot\frac{11}{21} = 81.$$

$$x^2 + \frac{81}{49}x^2 + \frac{66}{49}x^2 = 4x^2 = 81.$$

Solving yields $x = \frac{9}{2}$, so the answer is $9+2 = \boxed{011}$. □

Example 2.7 (2009 AIME I #15)

In triangle ABC, $AB = 10$, $BC = 14$, and $CA = 16$. Let D be a point in the interior of $\overline{BC}$. Let I_B and I_C denote the incenters of triangles ABD and ACD, respectively. The circumcircles of triangles BI_BD and CI_CD meet at distinct points P and D. The maximum possible area of $\triangle BPC$ can be expressed in the form $a - b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.



Solution. Since BI_BDP and CI_CDP are cyclic, we have quite a few angle conditions. This motivates us to look at the angles of $\triangle BPC$. First, note that

$$\angle BI_BD = 180^\circ - \angle DBI_B - \angle BDI_B = 180^\circ - \frac{1}{2}\angle B - \frac{1}{2}\angle ADB = 90^\circ + \frac{1}{2}\angle BAD.$$

Similarly, $\angle CI_CD = 90^\circ + \frac{1}{2}\angle CAD$. By cyclic quadrilaterals BI_BDP and CI_CDP, $\angle DPB = 180^\circ - \angle BI_BD = 90^\circ - \frac{1}{2}\angle BAD$ and $\angle DPC = 180^\circ - \angle BI_BD = 90^\circ - \frac{1}{2}\angle CAD$. Therefore,

$$\angle BPC = \angle DPB + \angle DPC = 180^\circ - \frac{1}{2}\angle BAD - \frac{1}{2}\angle CAD = 180^\circ - \frac{1}{2}\angle A.$$

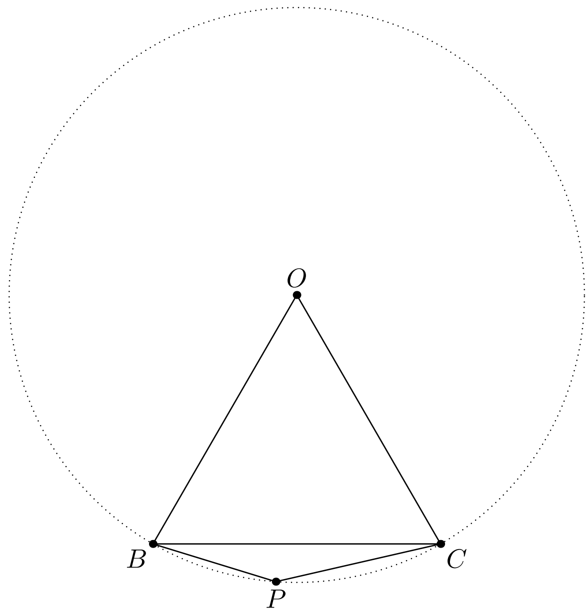
We can find $\angle A$ with the Law of Cosines on $\triangle ABC$. We have

$$\cos A = \frac{10^2 + 16^2 - 14^2}{2 \cdot 10 \cdot 16} = \frac{1}{2}.$$

Therefore, $A = 60^\circ$, so $\angle BPC = 150^\circ$. Therefore, $\triangle BPC$ has both a fixed side BC and a fixed angle $\angle BPC$. The locus of points P must be the arc BC of some circle. However, we can actually precisely state what this circle is. Since $\angle BPC = 150^\circ$, the minor arc BC must be equal to $360^\circ - 2 \cdot 150^\circ = 60^\circ$. Also, the radius of this circle is the circumradius of BPC, which from the Law of Sines is

$$\frac{BC}{2\sin\angle BPC} = \frac{14}{1} = 14.$$

Therefore, we see the center of this circle is the point O such that OBC is equilateral. We can now remove all the other points from the diagram, we just have



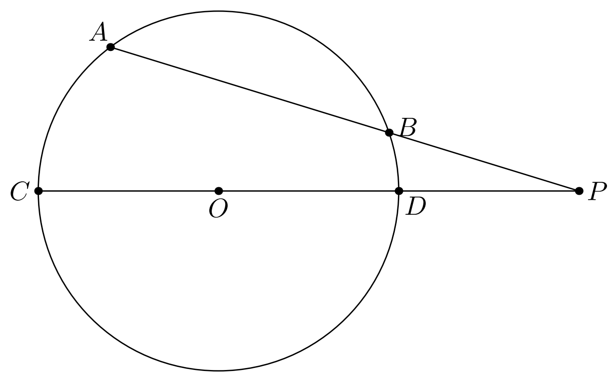
Now, we can find the area of $\triangle BPC$ with base times height. The base known to be $BC = 14$. To maximize the area, we have to maximize the height. The maximum height clearly occurs at the midpoint of minor arc BC. At this point, the height is $14 - 7\sqrt{3}$. Therefore, the area is

$$[\triangle BPC] = \frac{1}{2} \cdot 14 \cdot (14 - 7\sqrt{3}) = 98 - 49\sqrt{3}.$$

The final answer is $98 + 49 + 3 = \boxed{150}$. $\square$

§3 Power of a Point and Radical Axes

Recall Power of a Point: If A, B, C, D are four points on a circle and $\overline{AB}$ and $\overline{CD}$ intersect at P, then $PA \cdot PB = PC \cdot PD$. That is, if a line through P meets a fixed circle ω at A and B, the value of $PA \cdot PB$ is independent of the line.



How else can we approach this quantity? Let's choose the diameter of the circle $\overline{CD}$ containing P. If the center of the circle is O and its radius is R, and P is outside the circle, then

$$PC \cdot PD = (PO + R)(PO - R) = OP^2 - R^2.$$

If P is inside the circle, check that the product is instead $R^2 - OP^2$. We call $OP^2 - R^2$ the *power* of P with respect to ω. Note that the power is negative if P is inside the circle and positive if P is outside of the circle.

We now introduce a useful definition regarding the power of a point with respect to two circles:

Theorem 3.1 (Definition of a radical axis)

Let ω_1 and ω_2 be two nonconcentric circles in the plane, with centers O_1 and O_2. Then the set of all points P in the plane such that the power of P with respect of ω_1 equals the power of P with respect to ω_2 is a line called the *radical axis* of ω_1 and ω_2. This radical axis is perpendicular to $\overline{O_1O_2}$.

Proof. Coordinates allow for a simple proof. Without loss of generality, assume ω_1 is the unit circle centered at the origin. Let ω_2 be centered at $(a, 0)$ with radius r. Then the power of (x, y) with respect to ω_1 is

$$O_1P^2 - R_1^2 = x^2 + y^2 - 1.$$

The power of (x, y) with respect to ω_2 is

$$O_2^2P^2 - R_2^2 = (x-a)^2 + y^2 - r^2.$$

These two powers are equal only if

$$x^2 + y^2 - 1 = (x-a)^2 + y^2 - r^2 \iff x = \frac{a^2 - r^2 + 1}{2a}$$

which is the equation of a line perpendicular to $\overline{O_1O_2}$. $\square$

Corollary 3.2

If ω_1 and ω_2 intersect then their radical axis is their common chord.

Proof. Note that an intersection of the circles has zero power with respect to both of them and hence lies on their radical axis. So the radical axis is the line through the two intersections of the circles. $\square$

One powerful result about radical axes is:

Theorem 3.3 (Radical Axis Theorem)

Let ω_1, ω_2, ω_3 be pairwise nonconcentric circles. Let ℓ_1 be the radical axis of ω_2 and ω_3 and define ℓ_2 and ℓ_3 similarly. Then ℓ_1, ℓ_2, and ℓ_3 are either concurrent or all parallel.

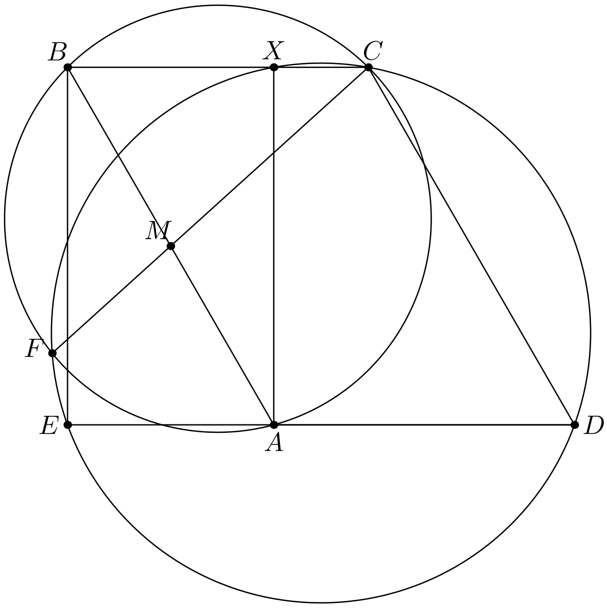
Proof. Suppose that ℓ_1 and ℓ_2 meet at a point P. Then since $P \in \ell_1$, P has equal powers with respect to ω_2 and ω_3. Since $P \in \ell_2$, P has equal powers with respect to ω_3 and ω_1. Thus P has equal powers with respect to ω_1 and ω_2, meaning it lies on ℓ_3.

This logic shows that if any two radical axes intersect then all three concur. Thus the only way for them to not concur is for all three of them to be parallel. Since the radical axis of two circles is perpendicular to the line joining their centers, it follows that in this case the centers of the three circles are collinear. $\square$

Definition 3.4. If the three radical axes concur, then their point of concurrency is called the *radical center* of the three circles. It has equal power with respect to all three circles.

Example 3.5 (2015 ARML Team # 10)

Let ABCD be a parallelogram with $m\angle A > 90°$. Point E lies on $\overrightarrow{DA}$ such that $\overline{BE} \perp \overline{AD}$. The circumcircles of $\triangle ABC$ and $\triangle CDE$ intersect at points F and C. Given that $AD = 35$, $DC = 48$, and $CF = 50$, compute all possible values of AC.



Solution. We know the dimensions of the parallelogram, but it seems quite hard to deal with the point F. It's the intersection of two circles, but there's not a lot going on between these two circles. However, we do notice that $\overline{CF}$ is the radical axis of the two circles, as it is their common chord. This suggests to us that we might want to search for a third circle so that we can use the Radical Axis Theorem. If we choose this third circle wisely, we can get additional information on the line $\overline{CF}$.

We want our third circle to intersect (ABC) and (CDE) at simple points, so that the radical axes are easy to work with. We spot one candidate: (ABE). Indeed, it meets (ABC) at A and B. One of its intersections with (CDE) is E, and we see that its other intersection is nice too! Indeed, it seems to be the intersection of (CDE) with $\overline{BC}$. Let's investigate this more.

Let X be the foot from A to $\overline{BC}$; it looks like this is our intersection point. Indeed, since $\overline{BE} \perp \overline{AD}$ and $ABCD$ is a parallelogram we see that $AEBX$ is a rectangle, so X lies on (ABE). The rectangle also implies $EX = AB = CD$. Since $\overline{CX} \parallel \overline{DE}$ it follows that $CDEX$ is an isosceles trapezoid, which is cyclic. Thus X lies on (CDE) as well.

Using the radical axis theorem on (ABC), (CDE), and (ABE), we get that $\overline{CF}$, $\overline{AB}$, and $\overline{EX}$ concur. But $\overline{AB}$ and $\overline{EX}$ bisect each other, so we know that $\overline{CF}$ bisects $\overline{AB}$.

Let M be the midpoint of $\overline{BC}$. Then the given lengths imply $AM = BM = 24$, $BC = 35$, and $CF = 50$. By Power of a Point

$$MA \cdot MB = MC \cdot MF \implies 24 \cdot 24 = CM(50 - CM).$$

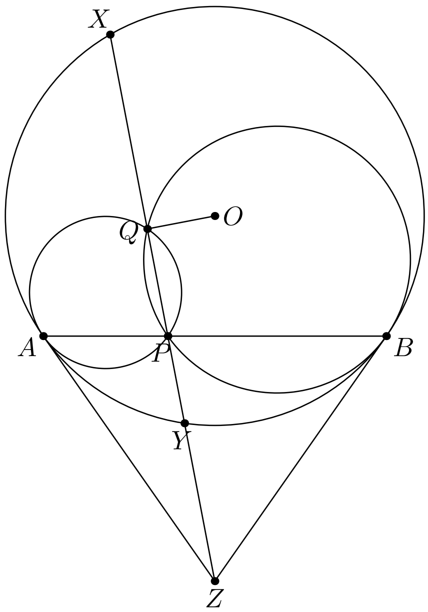
Solving this gives $CM = 32, 18$. Using the median length formula on $\triangle ABC$ gives

$$4CM^2 = 2AC^2 + 2BC^2 - AB^2 \implies AC^2 = \frac{4CM^2 + AB^2 - 2BC^2}{2}.$$

This gives the possible values of AC as $\boxed{5\sqrt{23}, 5\sqrt{79}}$. □

Example 3.6 (2019 AIME I # 15)

Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. We see immediately that $\overline{PQ}$ is the radical axis of ω_1 and ω_2. How can we use this information in a useful manner?

The answer lies in the tangencies of the circles. One very useful property of tangent circles is that they have a common tangent line at their point of tangency. So let's draw the common tangent of ω and ω_1, and also the common tangent of ω and ω_2. Suppose that these tangents meet at Z, so that $\overline{ZA}$ is tangent to ω and ω_1, and $\overline{ZB}$ is tangent to ω and ω_2.

Now by equal tangents, $ZA = ZB$. However, ZA^2 is the power of Z with respect to ω_1, and ZB^2 is the power of Z with respect to ω_2. So Z has the same power with respect to the two circles, and hence it lies on their radical axis $\overline{PQ}$!

This is a useful observation, but it doesn't give us enough information to solve the problem. Looking at the diagram, we notice a possible fact. Is $AQBZ$ cyclic? We can quickly confirm our suspicions with angle chasing. Since $\overline{ZA}$ is tangent to ω_1, we have $\angle AQP = \angle PAZ$. And since $\triangle ZAB$ is isoceles, we have $\angle PAZ = \angle ZBA$. Thus

$\angle AQP = \angle ZBA$, implying the cyclicity.

Finally, we recall that if O is the center of ω, then $\angle OAZ = \angle OBZ = 90^\circ$ by the tangent lines. Thus the circumcircle of (AZB) has diameter $\overline{OZ}$. But Q lies on this circle, so $\angle OQZ = 90^\circ$. This implies that Q is actually the midpoint of $\overline{XY}$.

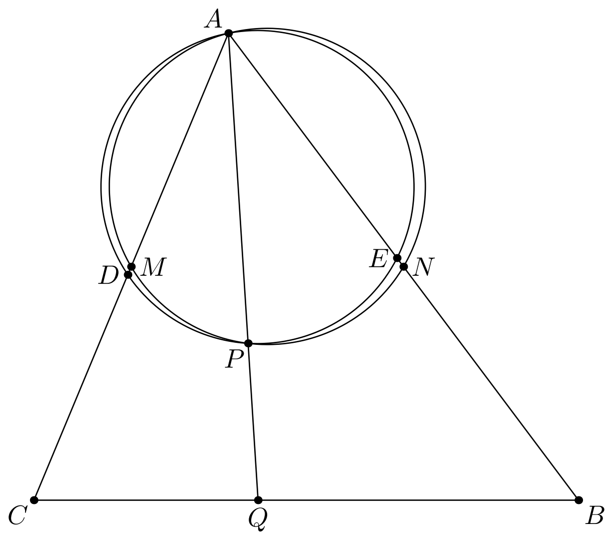
Armed with this simplification, we can now jump into the computation. We have $XQ = YQ = \frac{11}{2}$, so that $PY = \frac{11}{2} - PQ$. So by Power of a Point

$$PA \cdot PB = PX \cdot PY \implies 5 \cdot 3 = \left(\frac{11}{2} + PQ\right)\left(\frac{11}{2} - PQ\right).$$

Solving this equation gives $PQ^2 = \frac{61}{4}$ so the answer is $\boxed{65}$. □

Example 3.7 (2010 AIME II #15)

In triangle ABC, $AC = 13, BC = 14$, and $AB = 15$. Points M and D lie on AC with $AM = MC$ and $\angle ABD = \angle DBC$. Points N and E lie on AB with $AN = NB$ and $\angle ACE = \angle ECB$. Let P be the point, other than A, of intersection of the circumcircles of $\triangle AMN$ and $\triangle ADE$. Ray AP meets BC at Q. The ratio $\frac{BQ}{CQ}$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m - n$.



Solution. In this solution, we introduce a new concept, known as linearity of power. Roughly speaking, if ω_1, ω_2 are two circles, then the function $f(P) = \text{Pow}_{\omega_1}(P) - \text{Pow}_{\omega_2}(P)$ behaves linearly. We formalize this below:

Theorem 3.8 (Linearity of Power)

If ω_1, ω_2 are circles in the plane, then the function $f(P) = \text{Pow}_{\omega_1}(P) - \text{Pow}_{\omega_2}(P)$ is a linear function in P.

Proof. We can use coordinates to prove this in a similar manner to how we showed the existence of the radical axis.

In particular, if we have that $f(P)$ is a linear function in x_P and y_P, its coordinates, then it must be a linear function in P as well. So, denote the center of ω_1 as (x_1, y_1), the center of ω_2 as (x_2, y_2), and their radii as r_1, r_2 respectively. We have

$$f((x,y)) = (x-x_1)^2 + (y-y_1)^2 - r_1^2 - (x-x_2)^2 - (y-y_2)^2 + r_2^2$$

Note that the x^2 and y^2 terms cancel out, so $f(P)$ is indeed a linear function in its coordinates, as desired. $\square$

The main power of linearity of power is that we can take weighted averages of f. As a simple example, if we have points X, Y with midpoint Z, then $\frac{f(X)+f(Y)}{2} = f(Z)$ by linearity of power. Let's see how to apply it to this question.

Define $f(X)$ to be $\text{Pow}_{(ADE)}(X) - \text{Pow}_{(AMN)}(X)$. Then, by linearity of power, we should have that

$$\frac{QB \cdot f(C) + QC \cdot f(B)}{BC} = f(Q)$$

But, Q lies on the radical axis of the two circles, so $f(Q) = 0$. Hence,

$$QB \cdot f(C) + QC \cdot f(B) = 0 \implies \frac{BQ}{CQ} = -\frac{f(B)}{f(C)}$$

Luckily, the powers of B, C with respect two the two circles is relatively easy to calculate. If we consider B, angle bisector theorem gives $BE = \frac{14}{14+13} \cdot 15 = \frac{70}{9}$. Hence,

$$P(B) = BE \cdot BA - BN \cdot BA = 15\left(\frac{70}{9} - \frac{15}{2}\right) = \frac{25}{6}$$

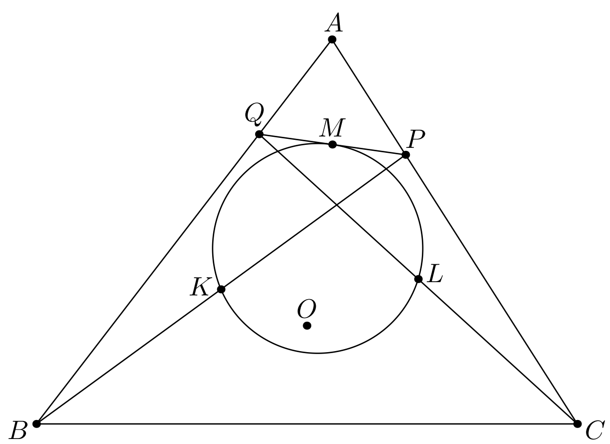
Similarly, $CD = \frac{14}{14+15} \cdot 13 = \frac{182}{29}$, so

$$P(C) = CD \cdot CA - CM \cdot CA = 13\left(\frac{182}{29} - \frac{13}{2}\right) = \frac{-169}{58}$$

So, $\frac{BQ}{CQ} = \frac{25/6}{169/58} = \frac{725}{507}$, and our answer is $725 - 507 = \boxed{218}$. $\square$

Example 3.9 (IMO 2009/2)

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.



Proof. One may be tempted to try to show OPQ is isosceles or something like that. However, note that O is the center of (ABC), so $OP = OQ$ actually means that P, Q have the same power with respect to (ABC). So, it suffices to show that $AQ \cdot BQ = AP \cdot PC$.

Now, note that $KM \parallel BQ$, $ML \parallel PC$ since they are midlines, so we have

$$\angle MKL = \angle LMP = \angle QPA$$

and similarly $\angle MLK = \angle AQP$, so $AQP \sim MLK$ and $\frac{AQ}{AP} = \frac{ML}{MK}$.

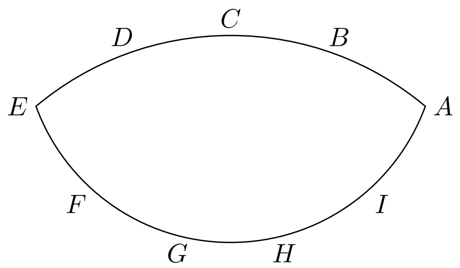
Also, $BQ = 2MK$ and $PC = 2ML$, so $\frac{BQ}{PC} = \frac{MK}{ML}$. Hence,

$$\frac{AP}{AQ} \cdot \frac{PC}{BQ} = \frac{MK}{ML} \cdot \frac{ML}{MK} = 1 \implies AQ \cdot BQ = AP \cdot CP$$

as desired. $\square$

§4 Problems

Problem 4.1 (2015 AIME I #6)**.** Point A, B, C, D, and E are equally spaced on a minor arc of a circle. Points E, F, G, H, I and A are equally spaced on a minor arc of a second circle with center C as shown in the figure below. The angle $\angle ABD$ exceeds $\angle AHG$ by 12°. Find the degree measure of $\angle BAG$.



Problem 4.2 (2011 AIME I #4)**.** In triangle ABC, $AB = 125$, $AC = 117$, and $BC = 120$. The angle bisector of angle A intersects $\overline{BC}$ at point L, and the angle bisector of angle B intersects $\overline{AC}$ at point K. Let M and N be the feet of the perpendiculars from C to $\overline{BK}$ and $\overline{AL}$, respectively. Find MN.

Problem 4.3 (orthocenter is incenter of orthic triangle)**.** Let ABC be a triangle, and let D, E, and F be the feet of the altitudes from A, B, and C, respectively. Let H be the orthocenter of ABC. Prove that H is the incenter of DEF.

Problem 4.4 (2019 HMMT Geometry #5)**.** Isosceles triangle ABC with $AB = AC$ is inscribed in a unit circle Ω with center O. Point D is the reflection of C across AB. Given that $DO = \sqrt{3}$, find the area of triangle ABC.

Problem 4.5 (2019 AIME I #6)**.** In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

Problem 4.6 (2016 AIME I #6)**.** In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect AB at L. The line through C and L intersects the circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, then $IC = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problem 4.7 (Iran Incenter Lemma)**.** In triangle ABC, D, E and F are the points of tangency of incircle with the center of I to BC, CA and AB respectively. Let BI meet EF are L, and let CI meet EF at K.

(a) Show that $\angle BKC = \angle BLC = 90^\circ$.

(b) Show that I is the incenter of DKL.

Problem 4.8 (2016 HMMT Geometry #5)**.** Nine pairwise noncongruent circles are drawn in the plane such that any two circles intersect twice. For each pair of circles, we draw the line through these two points, for a total of $\binom{9}{2} = 36$ lines. Assume that all 36 lines drawn are distinct. What is the maximum possible number of points which lie on at least two of the drawn lines?

Problem 4.9 (2020 CMIMC Geometry # 7)**.** In triangle ABC, points D, E, and F are on sides BC, CA, and AB respectively, such that $BF = BD = CD = CE = 5$ and $AE - AF = 3$. Let I be the incenter of ABC. The circumcircles of BFI and CEI intersect at $X \neq I$. Find the length of DX.

Problem 4.10 (IMO 2013/4)**.** Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Problem 4.11 (USA TST 2011/4)**.** Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $OA \perp RA$.

Problem 4.12 (2016 PUMaC Geometry # 8)**.** Let $\triangle ABC$ have side lengths $AB = 4$, $BC = 6$, $CA = 5$. Let M be the midpoint of MC and let P be the point on the circumcircle of $\triangle ABC$ such that $\angle MPA = 90°$. Let D be the foot of the altitude from B to AC, and let E be the foot of the altitude from C to AB. Let PD and PE intersect line BC at X and Y, respectively. COmpute the square of the area of $\triangle AXY$.

Problem 4.13 (2019 ARML Team # 10)**.** Triangle ABC is inscribed in circle ω. The tangents to ω at B and C meet at the point T. The tangent to ω at A intersects the perpendicular bisector of $\overline{AT}$ at point P. Given that $AB = 14$, $AC = 30$, and $BC = 40$, compute $[PBC]$.

Problem 4.14 (2016 AIME I # 15)**.** Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

Problem 4.15 (2020 HMMT Team # 3)**.** Let ABC be a triangle inscribed in a circle ω and ℓ be the tangent to ω at A. The line through B parallel to AC meets ℓ at P, and the line through C parallel to AB meets ℓ at Q. The circumcircles of ABP and ACQ meet at $S \neq A$. Prove that AS bisects BC.

Problem 4.16 (USAJMO 2012/1)**.** Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).

Problem 4.17 (USAJMO 2013/3)**.** In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ = BP/PC$.

Problem 4.18 (APMO 2020/1)**.** Let Γ be the circumcircle of $\triangle ABC$. Let D be a point on the side BC. The tangent to Γ at A intersects the parallel line to BA through D at point E. The segment CE intersects Γ again at F. Suppose B, D, F, E are concyclic. Prove that AC, BF, DE are concurrent.

Problem 4.19. Let circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with centers O_1 and O_2 intersect at A and B. The common tangent closer to A touches $\mathcal{C}_1$ at M and $\mathcal{C}_2$ at N. Line MA meets $\mathcal{C}_2$ again at Q and line NA meets $\mathcal{C}_1$ again at P. F is the intersection of NQ and MP, and the circumcircles of FMN and FPQ intersect again at E. Prove that EF, MN, and O_1O_2 are concurrent.

Problem 4.20 (Shortlist 2013 G4)**.** Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.